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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXXI.

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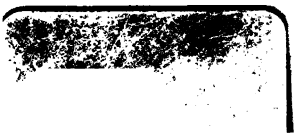

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WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

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EDITED BY

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5603. (C. Leudesdorf, M.A.)—Within a given circle a point is taken at random, and with this as centre and any radius taken at random a circle is described so as to lie wholly within the first. If the first circle be now reciprocated with regard to the second, find the average area of the resulting ellipse 119

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9845. (Professor Bordage.)—Construct a triangle right-angled at A, knowing (1) the ratio of the bisectors of the angles B and C, limited at their point of intersection 0; (2) the surface m 64

10500. (D. Biddle.)—Prove that

$$\pi = \left(366\tfrac{1}{3} + \Sigma_1^{\infty}\frac{9}{n^2(n+1)^2\{3+16n(n+1)\}}\right)^{\frac{1}{2}} - 16.$$

[The materials for solving this Question are contained in Vol. L., App. I. The formula is a convenient one for finding π] 70

13374 & 13393. (R. Chartres.)—If two points P and Q be taken at random in the sides AB, AC of a triangle, find (1) the mean value of the area of the circumcircle of PAQ, and (2) the nth power of the triangle PQR if R be a random point in BC 40

13554. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Sum to n terms the series $5 + 22 + 57 + 116 + 205 + \ldots$ 65

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13567. (R. W. D. Christie.)—Prove that an infinite number of integral values of a, b, A, B can be found to satisfy the equation

$$(3a^2 + 2ab + 2b^2)^2 \equiv (3A^2 + 2AB + 2B^2) + 2 \quad \ldots\ldots\ldots\ldots \quad 84$$

13571. (Rev. W. Allen Whitworth, M.A.)—A stick 18 inches long is divided at random into three parts. John is to have the longest, Tom the next, and Sam the shortest. Determine their expectations 81

13588. (H. W. Curjel, M.A.)—Prove, by Euclid, Book I., that any triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the triangles, equiangular to it, described on the other two sides 46

13635. (The late Professor Wolstenholme, M.A., D.Sc.)—On the sides of a given triangle ABC are erected equilateral triangles aBC, a'BC, &c.; A, a being on the same side of BC, A, a' on opposite sides of BC, and similarly for b, b', c, c'. It is well known that $Aa = Bb = Cc$, $Aa' = Bb' = Cc'$; but prove the following properties:—(1) Aa, Bb, Cc are each equal to three times the difference of the semiaxes of the maximum ellipse inscribed in the triangle ABC. (2) Aa', Bb', Cc' are each equal to three times the sum of the semiaxes of the maximum inscribed ellipse; and to the altitude of the maximum equilateral triangle circumscribed about the given triangle. (3) The straight lines bisecting the angles between the three pairs Aa, Aa', Bb, Bb', Cc, Cc' are all parallel to the axes of the maximum inscribed ellipse. In fact, if G be the centre of this ellipse, A', B', C' the points of contact (the mid-points of the sides), and if on the straight line drawn through A' at right angles to BC be taken points α, α', such that $\alpha\alpha'$ is bisected in A', and is equal to the diameter parallel to BC, Gα is parallel to Aa and $= \frac{1}{3}Aa$; Gα' is parallel to Aa' and $= \frac{1}{3}Aa'$; and, by the well-known construction for the axes of an ellipse from a given pair of conjugate diameters, Gα is equal to the difference, and Gα' to the sum of the semiaxes, and Gα, Gα' are equally inclined to the axes 101

13642. (H. W. Curjel, M.A.)—Find the equation to the surface described by a point the sum of whose distances from three fixed points is constant 84

13651. (R. F. Muirhead.)—In the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{matrix}$$

there are 40 determinants which can be formed by suppressing 2 columns and 1 row. Show that six of these can be chosen so that their vanishing involves the vanishing of all the others. Extend this to an array of p columns and q rows, and determinants of the order r, where r is less than either p or q 121

13678. (Rev. T. Mitcheson, B.A.)—In the triangle ABC, let AB be produced to F, CA to E, and BC to D; bisect the exterior angles by lines intersecting at α, β, and γ, thus forming the triangles $\alpha\beta\gamma$, BαC, AβC, and AγB. These triangles are equiangular, each angle being the arithmetical mean of the corresponding pair of the angles of ABC 48

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13712. (W. C. Stanham, B.A.)—If the centroid of a triangle is a fixed point, (*a*) its angular points lie on a fixed conic if its sides touch a fixed conic, (*b*) its sides touch a fixed conic if its angular points lie on a fixed conic 83

13737. (The late K. A. H. Fischer.)—Solve the equations:—

$$x_1+x_2+x_3+x_4+x_5=1; \quad x_1y_1+x_2y_2+x_3y_3+x_4y_4+x_5y_5=0;$$
$$x_1y_1^2+\ldots=\tfrac{1}{3}; \quad x_1y_1^3+\ldots=0; \quad x_1y_1^4+\ldots=\tfrac{1}{3}; \quad x_1y_1^5+\ldots=0;$$
$$x_1y_1^6+\ldots=\tfrac{1}{7}; \quad x_1y_1^7+\ldots=0; \quad x_1y_1^8+\ldots=\tfrac{1}{9}; \quad x_1y_1^9+\ldots=0$$

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13751. (Professor Sanjána.)—The equation of the ellipse touching the sides of a triangle at the mid-points is $\Sigma\sqrt{(a\alpha)}=0$. Prove that the centre of this ellipse is at the centroid of the triangle, and that one focus of it is the symmedian point of its own pedal triangle with regard to the original triangle. [This seems the easiest way of obtaining the fundamental property of Steiner's foci.] 47

13778. (Professor Crofton, F.R.S.)—If Δ is a triangular number, $9\Delta+1$ is also one; also all triangular numbers, not multiples of 3, are of the form $9\Delta+1$. If Δ, Δ' are any two triangular numbers, $4(\Delta+\Delta')+1$ is the sum of two squares. If $4n+1$ is a prime number, $n=\Delta+\Delta'$.. 69

13783. (Professor Umes Chandra Ghosh.)—Prove that

$$\int_0^\infty \frac{\sin(n+1)x\,.\,\sin nx}{x\,.\,\sin x}\,dx=\tfrac{1}{2}(n\pi) \ldots\ldots 53$$

13796. (C. E. Bickmore, M.A.)—Solve the difference-equations

(1) $u_n^4+1=(u_{n-1}u_n+1)(u_nu_{n+1}+1)$,

(2) $u_n^4-u_n^2+1=(u_{n-1}u_n+1)(u_nu_{n+1}+1)$ 53

13802. (R. Chartres.)—Sum to infinity

$$1+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1.3}{2.4}+\frac{1}{4}\cdot\frac{1.3.5}{2.4.6}+\&c. \ldots\ldots 50$$

13816. (H. W. Curjel, M.A.)—Find the form of the function f such that $f(a+x)+(b+x)-f(a+x)f(b+x)=f(c+x)$ where c depends only upon a and b. Also determine c as a function of a and b 51

13822. (Robert W. D. Christie.)—Prove the following curious theorem, which is true for primes only. (Is there any other, except Wilson's?):— $(2s)^p\{(2s)^p+1\}\equiv 1 \pmod p$, where $s=\sin 18°$ and p any odd prime whatever 82

13824. (R. Knowles, B.A.)—P, P′ are any two points on a parabola: the tangent at P meets the diameter through P′ in Q. Prove that the tangent at P′ bisects PQ. [See Quest. 13372, and Besant, *Conics*, Prop. XVII., § 45] 54

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13829. (R. Chartres.)—If from the vertex A of a triangle there be drawn to the base the altitude AD, the bisector of the angle AE, and the median AF, then, if DE = 3EF, the sides of the triangle will be in arithmetical progression 57

13840. (Professor A. Droz-Farny.)—On considère une circonférence fixe C et une circonférence variable C′ qui reste constamment tangente à deux droites donnés. Quelle est l'enveloppe de l'axe radical des circonférences C et C′? 52

13858. (Robert W. D. Christie.)—Prove the following:—Where P is any prime whatever, 4P + 5 equals one, two, or three integral squares. Distinguish the three cases 54

13863. (The late R. Potts, LL.D.)—In the diagram (Potts' *Euclid*) of Euc. I. 2, with centre D and radius DA, describe a circle intersecting the circle CGH in the point M. Join DM, and produce it to meet the circumference of the circle GKL in N; then MN shall be equal to AL. What is the condition that this construction may be possible?..... 84

13866. (Professor Neuberg.) — Un paraboloïde hyperbolique est déterminé par quatre points A, B, C, D et par deux plans P, Q auxquels les génératrices des deux modes sont respectivement parallèles. Construire les génératrices qui passent par A, B, C, D. Trouver l'équation du paraboloïde 40

13871. (W. J. C. Miller, B.A.)—In order to determine the position of a rock, which was covered at high water, the angles subtended at it by the distances between three headlands were observed, and found to be 151° 2′ 42″, 133° 25′ 57″, 75° 31′ 21″; and it was known, from a previous survey, that the distances which subtended these angles were respectively 9870, 10857, 8883 yards. Find the distance of the rock from each of the headlands; and show the numerical calculation 88

13875. (B. N. Cama, M.A.)—P is any point on an ellipse, foci S and H. Obtain the differential relation

$$\frac{d \log \mathrm{SP}}{d\theta} = \frac{d \log \mathrm{HP}}{d\theta'},$$

where $\theta = \angle \mathrm{PSH}$, and $\theta' = \angle \mathrm{PHS}$.

Hence or otherwise deduce that

$$\frac{1}{\mathrm{SP}}\left\{e^2 . \mathrm{SP}^4 - l^2\left(\frac{d\mathrm{SP}}{d\theta}\right)^2\right\}^{\frac{1}{2}} + \frac{1}{\mathrm{HP}}\left\{e^2 . \mathrm{HP}^4 - l^2\left(\frac{d\mathrm{HP}}{d\theta'}\right)^2\right\}^{\frac{1}{2}}$$

is constant for all positions of P, $2l$ being the length of the latus rectum, and e the eccentricity. Is any geometrical solution possible?..... 37

13876. (B. H. Steede, M.A.)—If r be radius vector to any point P on a curve, ρ the radius of curvature at P, r' and ρ' the corresponding values for corresponding point on inverse curve, and ρ'' the radius of curvature of reciprocal curve at point of contact of tangent corresponding to point P, k radius of circle of reciprocation, prove

$$\frac{r}{\rho} + \frac{r'}{\rho'} = \frac{2k^{\frac{1}{2}}}{(\rho\rho'')^{\frac{1}{2}}} \quad \ldots\ldots\ldots\

13878. (R. F. Davis, M.A.)—AD, BE, CF are the medians of a triangle ABC co-intersecting in G, the centroid; and H is the orthocentre. Upon DA a point α is taken, so that $DA \cdot D\alpha = DB^2 = DC^2$; similarly for β, γ. Prove that the triangle $\alpha\beta\gamma$ has its sides proportional to the medians of ABC, and that GH is a diameter of its circumcircle 45

13879. (J. Brill, M.A.)—Verify that the six quantities $x, y, z, \xi, \eta, \zeta$ can be eliminated from the six equations

$$a + dx + e\xi = 0, \quad b + dy + e\eta = 0, \quad c + dz + e\zeta = 0,$$
$$f_1 + g_2 z - g_3 y - h_2\zeta + h_3\eta + k(y\zeta - z\eta) = 0,$$
$$f_2 + g_3 x - g_1 z - h_3\xi + h_1\zeta + k(z\xi - x\zeta) = 0,$$
$$f_3 + g_1 y - g_2 x - h_1\eta + h_2\xi + k(x\eta - y\xi) = 0,$$

and obtain the result of elimination 58

13880. (Rev. W. Allen Whitworth, M.A.)—Find three numbers (each less than 10,000) whose squares are in arithmetical progression, so that the ratio of the greatest to the least may be (i.) as great as possible, and (ii.) as small as possible 56

13886. (John F. Wilkinson, B.A.)—If equilateral triangles be described on the sides of a right-angled triangle whose hypotenuse is given in magnitude, the locus of the middle point of the line joining their vertices is a circle. (By First Book Euclid, Casey's *Sequel to Euclid*). 28

13887. (J. J. Barniville, B.A.)—Prove that

$$\frac{19}{1^3 \cdot 2^3} + \frac{127}{3^3 \cdot 4^3} + \frac{331}{5^3 \cdot 6^3} + \frac{631}{7^3 \cdot 8^3} + \ldots = \frac{\pi^2}{4} \quad \ldots\ldots 39$$

13889. (G. H. Hardy.)—Sum $1 - \frac{3}{5} + \frac{3}{9} - \frac{4}{13} + \ldots$ ad inf. 26

13892. (P. W. Flood.)—Inscribe a triangle in a given segment of a circle, having the sum of the perpendicular and segment of base a maximum 46

13898. (Professor Neuberg.)—Soient (A,B,C), (A′,B′,C′), (A″,B″,C″), (A‴, B‴, C‴) les angles de quatre triangles ABC, ABC′, ABC″, ABC‴ construits sur la même base AB. Démontrer que la condition nécessaire pour que les sommets C, C′, C″, C‴ appartiennent à une même circonférence est

$$\begin{vmatrix} \cot A & \cot B & \cot C & 1 \\ \cot A' & \cot B' & \cot C' & 1 \\ \cot A'' & \cot B'' & \cot C'' & 1 \\ \cot A''' & \cot B''' & \cot C''' & 1 \end{vmatrix} = 0 \quad \ldots\ldots 38$$

13904. (Professor Umes Chandra Ghosh.)—If ρ, ρ_n be the radii of curvature of a curve and its nth positive pedal at corresponding points whose radii vectores are r, r_n, show that $\frac{r_{n-1}}{\rho_n} + \frac{\rho_{n-1}}{r_{n-2}} = 2$. Hence show that $\frac{\rho_n}{r_{n-1}} = \frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\ldots - \frac{r_1\rho}{r^2}$ where $\frac{r_1\rho}{r^2}$ is the $(n+1)^{\text{th}}$ complete quotient 38

13908. (B. N. Cama, M.A.)—With a fixed point as focus and of given magnitude, a series of conics are described. Show that the extremities

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of all focal chords which are divided by the focus in a fixed ratio lie on either of two fixed circles. [Give a geometrical solution]..... 48

13921. (Rev. J. Cullen, B.A.)—Taking $\cos(A-\theta)$, $\cos(B-\theta)$, $\cos(C-\theta)$ as the coordinates and R′ the radius of a Tucker-circle, prove that (1) $R' = 2\Delta/\Sigma a \cos(A-\theta)$; (2) the circle intersects AB, AC in F and E′ (FE′ being antiparallel to BC), denoting the circle AFE′ by T_A and the circles BDF′, CED′ similarly obtained by T_B, T_C; then the centres of T_A, T_B, T_C lie on the perpendiculars from the vertices on the sides of ABC; (3) the radical axes of these three circles pass through fixed points; (4) their radical centre is the point

$$a \sin(A+\theta)/\sin 2A = \ldots = \ldots,$$

so that it lies on the inverse of the orthocentroidal axis, *i.e.*, a rectangular hyperbola circumscribing ABC 35

13927. (Professor Sanjána.)—Prove the identity

$$(x^{2n+1}-y^{2n+1})/(x-y) = (x^n + x^{n-1}y + \ldots + y^n)^2 - xy(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})^2;$$

and show how it may be employed to obtain factors of large numbers of the form $x^{4n+2}-1$ 63

13934. (B. H. Steede, M.A.)—Show that it follows immediately, by inversion, that any circle passing through the two points of intersection of one of the four axes of similitude and the orthogonal circle to three given circles cuts these circles at equal angles, and hence describe circles cutting three given circles at a given angle 67

13935. (J. Timmer.)—Construct a triangle ABC, having given the side BC, the angle B, and the length of the bisector of A. 29

13939. (C. E. Hillyer, M.A.)—AOB is the diameter of a circle of radius R; circles are described on OA and OB as diameters; a circle O_1 is then described, touching the last two externally and the original circle internally; a second circle O_2 touches O_1 and the circle on OA externally, and the original circle internally; and so on. Show that the angles AOO_2, AOO_3, AOO_4, ... are successive acute angles in descending order which have all their trigonometrical ratios rational, and the radius of the nth circle O_n is $\frac{R}{n^2+2}$. Also, if O be not the centre, but $OA = a$ and $OB = b$, then the diameter of the nth circle, similarly constructed, is $\frac{ab(a+b)}{n^2b^2 + a(a+b)}$. [See, regarding the second part, Quest. 7905, by R. Lachlan, B.A., Vol. LXI., p. 94] 59

13942. (R. F. Davis, M.A.)—Prove that from the relation

$$(x+y+z+p)(x^{-1}+y^{-1}+z^{-1}+q^{-1}) = \lambda$$

there may be deduced three equations:

$$\left.\begin{aligned} a(yz)^{-1} + byz + c + f(y+z) + g(y^{-1}+z^{-1}) + h(y/z + z/y) &= 0 \\ a(zx)^{-1} + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots &= 0 \\ a(xy)^{-1} + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots &= 0 \end{aligned}\right\} \text{(A)},$$

where $h^2 - ab - ch + fg = 0$. Conversely, ascertain whether from three equations of the form (A) the above relation may be inferred 55

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13950. (Ernest P. Barrett, B.A.)—If O be the point from which two common tangents are drawn to two orthogonal circles, whose points, A and C, of intersection are fixed, prove that, if the intersections of the polars of O with respect to the circles with the join of O to the centres are B and D, ABCD is a square 58

13955. (Professor Neuberg.)—On construit sur les côtés d'un triangle ABC trois triangles isoscèles semblables A'BC, B'CA, C'AB. Les droites BA', BC' coupent AC en B_1, B_2; les droites CB', CA' coupent BA en C_1, C_2; enfin, les droites AC', AB' coupent CB en A_1, A_2. Cela posé, les droites B_1C_2, C_1A_2, A_1B_2 enveloppent trois hyperboles, qui touchent deux des côtés du triangle ABC et ont pour asymptote le troisième côté; ces courbes ont pour tangente commune la droite de Lemoine de ABC... 25

13958. (Professor Sanjána, M.A.)—Resolve into prime factors:—
(1) 134^4+3^4, (2) 86^4+1, (3) 1786^4+9^4, (4) 3266^4+1 ... 72

13959. (Professor F. Rogel.)—Prove that, if

(1) $$K_p = q + 2\sum_{\nu=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}\nu}}{(\nu+1)!} B\left(\frac{1+q}{2}, \nu+1\right) \cdot \left(\frac{2\pi}{p}\right),$$

where $$q = (p-1)!, \quad p>3,$$

and B denotes the "Bernoullian function" of the $(\nu+1)$th order;

(2) $$L_p = -1 + \sum_{\nu=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu+2)}}{\nu!} E(r,\nu)\left(\frac{\pi}{2p}\right)^{\nu},$$

where $$r = \left[1.3.5\ldots(p-2)\right]^2, \quad p>3,$$

and E denotes the "Eulerian function" of the first species of the νth order. [F. Rogel, "Theorie der Euler'schen Function," *Sitzungsbericht der kgl. böhmischen Gesellschaft der Wissenschaften*, Prag, 1893, XXIII.];

(3) $$M_p = (-1)^{\frac{1}{2}(p-1)} \sum_{\nu=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu-1)}}{\nu!} E(q,\nu)\left(\frac{\pi}{2p}\right)^{\nu}, \quad p>3;$$

(4) $$N_p = \sum_{\nu=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu-1)}}{\nu!} E(s,\nu)\left(\frac{\pi}{2p}\right)^{\nu},$$

where $$s = \left[\left(\frac{p-1}{2}\right)!\right]^2, \quad p>3,$$

each of these series K_p, L_p, M_p, $N_p = 1$ if p be a prime number, and $= 0$ if p be complex 126

13960. (Professor Umes Chandra Ghosh).—Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{2n+1}\phi}{(1+\sin^2\phi)^{n+1}}\,\delta\phi = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left[\frac{1}{2}(n+1)\right]}{\Gamma\left[\frac{1}{2}(n+2)\right]} \ldots\ldots 70$$

13964. (G. H. Hardy.)—Prove

$$\int_0^{\frac{1}{2}\pi} \tanh^{-1}(p\tan^2 x)\,dx = \tan^{-1}\sqrt{p}\,\tanh^{-1}\sqrt{p} - \tfrac{1}{4}\pi\tanh^{-1}p,$$

$$\int_0^{\lambda} \log(1+\tan\lambda\tan x)\,dx = \lambda\log\sec\lambda,$$

$$\int_0^{\pi} (x\cot x)^2\,dx = \tfrac{2}{3}\int_0^1 \left(\frac{\sin^{-1}x}{x}\right)^3 dx = \pi\log 2 - \frac{\pi^3}{24} \ldots\ldots 86$$

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13966. (R. F. Davis, M.A.)—If B, C are given fixed points, and the angle BPC is constant, prove geometrically that the complete envelope of the inscribed circle of the triangle PBC is the base BC, and a parallel to a two-cusped epicycloid 79

13970. (C. E. Hillyer, M.A.)—F is a given point within an ellipse; CF meets the curve in P. Required to draw through F a straight line meeting the curve in Q, so that the angle between FQ and the tangent at Q is equal to that between CP and the tangent at P 79

13972. (Rev. W. Allen Whitworth, M.A.)—If a die of p faces be thrown n times, the expectation of the lowest throw is $1+k$ and the expectation of the highest is $p-k$, where

$$k = \{1^n + 2^n + 3^n + \ldots + (p-1)^n\} \div p^n \ldots\ldots 53$$

13973. (R. Knowles, B.A.)—$y^2 = 4ax$ is the equation to a parabola vertex A; on the tangent at A, AT is taken $= 4a$; from T, TP is drawn to touch the curve in P; the chord AQ is drawn at right angles to AP; the lines joining the pole of AP to Q and that of AQ to P intersect in O. Prove (1) that the equation to the polar of O is $21x + 6y + 16a = 0$; (2) PQ meets this polar on AT 63

13976. (Rev. J. Cullen, B.A.)—In Quest. 13921, if R_A, &c., are the radii of T_A, &c., prove that

$$(1)\ R_A \sin A = R_B \sin B = R_C \sin C = R' \sin\theta;$$

(2) the join of the centre of the circle DEF [*i.e.*, $a \sec(A-\theta) = \ldots = \ldots$ and the radical centre $a \sin(A+\theta)/\sin 2A = \ldots = \ldots$ passes through th orthocentre of ABC 87

13979. (R. Chartres.)—Show that

$$\int_0 \frac{x^{k-1} \cdot dx}{1+x} = \frac{\pi}{\sin k\pi}$$

without Demoivre's theorem or imaginaries ($k < 1$); evaluate

$$\int_0^\infty \frac{\log x \cdot dx}{1+x^3};$$

and sum the series $\frac{1}{1^2 \cdot 2^2} - \frac{3}{4^2 \cdot 5^2} + \frac{5}{7^2 \cdot 8^2} - \&c.$, *ad infinitum* 61

13987. (Professor Sanjána, M.A.)—The incircle of the triangle ABC touches the sides in X, Y, Z; $\theta, \theta_1, \theta_2, \theta_3$ are the Brocard angles of the triangles XYZ, AYZ, BZX, CXY, respectively. Prove that

$$\cot\theta_1 + \cot\theta_2 + \cot\theta_3 - \cot\theta = \cot\omega + \cot\theta = \text{cosec } A + \text{cosec } B + \text{cosec } C.$$

If XYZ be the excircle corresponding to BC, these formulæ take the form

$$-\cot\theta_1 + \cot\theta_2 + \cot\theta_3 - \cot\theta = -\cot\omega + \cot\theta$$
$$= -\text{cosec } A + \text{cosec } B + \text{cosec } C;$$

so also for the other excircles 64

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13994. (R. F. Davis, M.A.)—If ABC is an isosceles triangle D, the middle point of the base BC, and upon BA, CA points P, Q are taken such that $BP \cdot CQ = BD^2$, prove that the envelope of PQ is the circle whose centre is D and which touches AB, AC..... 41

13995. (Rev. W. A. Whitworth, M.A.)—Any number N is the sum of either one, two, or three triangular numbers. Show that the three cases may be discriminated as follows:—

(1) If $8N+1$ be a square, N is a triangular number.

(2) If $8N+1$ be not a square, and $4N+1$, when all square factors are cast out of it, contains no prime factor of the form $4n-1$, then N is the sum of *two* triangular numbers.

(3) In all other cases N is the sum of *three* triangular numbers ... 33

13996. (Rev. T. Roach, M.A.)—Sum to infinity

$$1-\tfrac{3}{7}+\tfrac{5}{13}-\tfrac{7}{19}+\tfrac{9}{25}-\tfrac{11}{31}+\ldots \text{ ..... } 128$$

13998. (J. J. Barniville, M.A.)—Prove that the series 1, 1, 2, 3, 5, ... contains multiples of every number. Prove that u_{mn} is divisible by u_n. Can it be shown that u_n is prime when n is prime?

[See *Nature*, May, 1897] 111

14002. (R. F. Muirhead.)—Find an algebraic function $\phi(x)$ which has the following properties connected with the series of numbers

$$3, 4, 7, 12, 19, \ldots, n^2+3.$$

(1) If applied to any one of these numbers except the first, it gives rise to the preceding one, *e.g.*, $\phi(12) = 7$, $\phi(7) = 4$; and applied to 3 it gives $\phi(3) = 4$.

(2) If applied to any number lying between two successive terms of the series (except 3 and 4), it gives a number in the next interval below; but, if applied to a number a lying between 3 and 4, the result is another number b lying between 3 and 4, such that $\phi(a) = b$ and $\phi(b) = a$.

(3) If applied to any of the numbers $3\frac{1}{4}$, $5\frac{1}{4}$, $9\frac{1}{4}$, ..., $n(n+1)+3\frac{1}{4}$ ... except the first, it reproduces the next below; and applied to $3\frac{1}{4}$ it makes $\phi(3\frac{1}{4}) = 3\frac{1}{4}$..... 85

14003. (R. Chartres.)—The attraction of an infinitely long cylindrical bar on an external point according to the law of the inverse square of the distance is $2\gamma\rho\kappa/p$ where κ = area of a section of the bar and p = the distance of the external point from the bar, γ and ρ being the usual constants. Explain the absurdity arising from this when the point is on the cylinder, and deduce the real value 67

14005. (G. H. Hardy.)—Evaluate the definite integrals

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(p\tan^2 t)\,dt, \quad \int_0^{\frac{1}{2}\pi} \tan^{-1}(p\sin^2 t)\,dt,$$

$$\int_0^a \frac{(a_0 a_1 a_2 \ldots a_n)(\log x, 1)^n}{\sqrt{(\log a - \log x)}}\,dx, \text{ and } \int_0^a \left(\frac{\sin}{\sinh}\right) \kappa \sqrt{(\log a - \log x)}\,dx$$

..... 111

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14006. (W. C. Stanham, B.A.)—Suggested by Quest. 13968.—A conic can be described having its asymptotes parallel to the sides AB, AC of a triangle ABC, passing through B and C, and having a focus at the point of contact with BC of the inscribed circle (or of an escribed circle) of ABC 33

14007. (Rev. T. Mitcheson, B.A.)—If

$$\frac{-x+y+z}{x+y-z}+\frac{x-y+z}{-x+y+z}+\frac{x+y-z}{x-y+z}=\frac{x}{z}+\frac{y}{x}+\frac{z}{y},$$

then $x+y+z=0$ 85

14009. (G. Hale Puckle, M.A.)—The angle between the positive sides of a pair of intersecting straight lines, represented by $\phi(xy)=0$, is greater, equal to, or less than a right angle, as $a+b$ is greater, equal to, or less than 0 32

14010. (Thomas T. Smith.)—A and B are two points on the same side of a given straight line XY such that the distance between A and B is less than the sum of their perpendicular distances from XY. Find the foci of two parabolas passing through A and B and having XY for their common directrix 86

14012. (Professor G. B. Mathews, M.A.)—Prove the identities—

(1) $a^5(b-c)+b^5(c-a)+c^5(a-b)+a(b-c)^5+b(c-a)^5+c(a-b)^5$
$+10abc(b-c)(c-a)(a-b)=0;$

(2) $a^7(b-c)+b^7(c-a)+c^7(a-b)+a(b-c)^7+b(c-a)^7+c(a-b)^7$
$+7abc(b-c)(c-a)(a-b)\{3(a^2+b^2+c^2)-2(bc+ca+ab)\}=0$
..... 68

14014. (Professor Sanjána, M.A.)—In the triangle ABC, AD is the median from A; AE and AE′ are the internal and external bisectors of the angle A. Prove that

$$\text{AD}\cos\text{DAE}=\tfrac{1}{2}(\text{AB}+\text{AC})\cos\tfrac{1}{2}\text{A},$$

and

$$\text{AD}\cos\text{DAE}'=\tfrac{1}{2}(\text{AB}\sim\text{AC})\sin\tfrac{1}{2}\text{A}.$$

Also apply these equalities to prove Feuerbach's theorem 77

14017. (D. Biddle.)—In the straight line AB a point K is taken, such that AK : KB = 1 : n, and another straight line BC, forming any angle with AB, is divided into r equal parts by points p_1, p_2,.... The parallelogram ABp_1Q being completed, and the necessary joins being effected, KQ intersects Ap_1, Ap_2, ... in k_1, k_2, ... Prove that

$$\text{A}k_1 : k_1p_1 = 1 : n+1, \quad \text{A}k_2 : k_2p_2 = 1 : n+2, \quad \ldots,$$

and

$$\text{A}k_r : k_rp_r = 1 : n+r.$$

[Of course, parallels to BC from k_1, k_2, ... divide AB in corresponding ratios. But it would be interesting if the above theorem could be utilized for the summation of the reciprocals of successive numbers]. 75

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14020. (Salutation.)—From a given point as vertex construct an equilateral triangle without describing any arc, but having a flat ruler with two of its edges at right angles and half the length of the required side set off on one of them, measured from their point of junction. [Not only is the ordinary joining of points postulated, but the apposition of either of the two specified points on the ruler to a point in the figure, as it grows, and of one of the edges to any line already drawn; also the drawing of a new line along the other edge, whilst the position of the ruler is maintained] 62

14022. (B. N. Cama, M.A.)—The pedal equation of a curve is given by $r^3 = pa^2$. In polars, this reduces to the lemniscate $r^2 = a^2 \sin 2\theta$. Can it be shown in a similar manner that the same equation may as well represent a fixed circle? 79

14023. (Rev. T. Roach, M.A.)—Point out the fallacy in the following statement:— $\tan^{-1}\frac{x}{1-x^2} + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x(1-x^2)}{1-4x^2+x^4}$.

Now, when $x = 1$, we have

$$\tan^{-1}\infty + \tan^{-1}\infty = \tan^{-1}0; \quad \text{therefore} \quad \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = 0 \ldots\ldots 78$$

14024. (G. Hale Puckle, M.A.)—If $\phi(xy) = 0$ represents a pair of real intersecting straight lines, then (with the conditions of Quest. 13535, No. 437) the equations

$$\alpha(ax+hy+g) + \beta(hx+by+f) = 0 \ldots\ldots (1),$$
$$\beta(ax+hy+g) - \alpha(hx+by+f) = 0 \ldots\ldots (2),$$

represent the bisectors of the angles between them; and (1) or (2) is the bisector of the acute angles, as $a+b >$ or < 0..... 32

14025. (Rev. W. Allen Whitworth, M.A.)—A number is expressed in the scale of notation whose radix is r, and its final digit is prime to r. Show that some power of the number will have its final digit unity. *E.g.*—In the common scale, if a number end in 3 or 7, its fourth power ends in 1; if it end in 9, its square ends in 1 32

14030. (H. MacColl, B.A.)—Show that in any triangle the bisector of the less of two unequal angles is greater than the bisector of the greater..... 31

14032. (Rev. T. Mitcheson, B.A.)—P and Q are the ends of conjugate semi-diameters of an ellipse, whose centre is C and foci H and S. The tangents meet in R, and M and N are the mid-points of HP and SQ; while CR meets PQ in T, MN in L. Show that the loci of M and N are ellipses whose common chord is equal to half the latus rectum of the original ellipse, the chord being bisected in C. The loci of L, T, and R are ellipses..... 41

14033. (R. Knowles, B.A.)—Prove that the $(r+1)$th term in the expansion of $\frac{1}{(1-x)^n}$ is $\frac{(r+1)(r+2)\ldots(r+n-1)}{n-1!}x^r$ 30

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14035. (H. J. Woodall, A.R.C.S.)—Is anything known about the theory of the following?—When is

$$a^p - 1 \equiv 0 \pmod{q^2},$$

it being given that a is not a power, and neither $a-1$ nor p divisible by q? *E.g.*, $(50,000)^6 - 1 \equiv 0 \pmod{11^2}$..... 43

14037. (I. Arnold.)—From a given point within a triangle, to draw three right lines trisecting the triangle..... 27

14040. (C. E. Bickmore, M.A. See Quest. 13998.)—If u_n be the nth term of the series 1, a, a^2+1, a^3+2a, ..., express u_n as a determinant, and prove the following:—

(1) $u_{2n} = u_n(u_{n+1} + u_{n-1})$, (2) $u_{2n+1} = u_{n+1}^2 + u_n^2$,

(3) $u_{3n} = u_n\{(u_{n+1} + u_{n-1})^2 - (-1)^n\}$, (4) the *proper* factors of u_n

(*i.e.*, those which are not factors of u_m, where n is a multiple of m) are of the form $Mn+1$ if they have (a^2+4) as a quadratic residue, and of the form $Mn-1$ if they have (a^2+4) as a quadratic non-residue 49

14042. (Professor G. B. Mathews, M.A.)—Prove that

$$a(b-c)^{6n+1} + b(c-a)^{6n+1} + c(a-b)^{6n+1}$$

is divisible by $(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$ for all positive integral values of n. Thus, when $n=1$, the quotient is

$$(a+b+c)(a^2+b^2+c^2+bc+ca+ab) - 18abc.$$

Is there any convenient way of finding the quotients for higher values of n?..... 57

14045. (Professor Sanjána.)—In a triangle ABC both AO = AI and BO = BI, where O and I are the circum- and in-centre, respectively. Determine the angles 81

14047. (Professor Umes Chandra Ghosh.)—Show that the volume generated by the revolution of a quadrant of an ellipse about a line through its centre whose inclination to the major axis is α, and which is outside the given quadrant, is

$$\tfrac{1}{3}\pi\left\{\frac{e^4\sin^2\alpha\cos^2\alpha - 1 + e^2}{(1-e^2\sin^2\alpha)^2}(a^3\cos^3\alpha + b^3\sin^3\alpha)\right.$$
$$+\frac{3b^2}{1-e^2\sin^2\alpha}(a\cos\alpha + b\sin\alpha) + \frac{2e^2\sin\alpha\cos\alpha}{(1-e^2\sin^2\alpha)^2}\frac{b}{a}(a^3\cos^3\alpha - b^3\sin^3\alpha)$$
$$\left.-\sin\alpha\cos\alpha(a^3\sin\alpha + b^3\cos\alpha)\right\}$$ 66

14048. (Professor Thomas Savage.)—"No integral values of x, y, z can be found to satisfy the equation $x^n + y^n = z^n$, if n be an integer greater than 2." (Fermat: W. W. Rouse Ball's *Short History of Mathematics*.) Prove the truth of this when x is any prime, y any positive integer not greater than x—the proof to hold for $n=2$ 100

14051. (R. Tucker, M.A.)—P is a point on the ordinate of an ellipse produced. PK is the perpendicular from P on the polar of P. Find the equation to the circular locus of K, and the envelope of this locus as the ordinate varies in position. Interpret the result..... 110

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14053. (Rev. W. A. Whitworth, M.A.)—Prove that Quest. 13630, Vol. LXIX., p. 47, is true if *any* square number be substituted for 4, *e.g.*,

$$1-9+9^2-9^3+9^4 = \square+\square = 73^2+24^2 = 63^2+44^2,$$

$$1-25+625 = \square+\square = 24^2+5^2;$$

and generally that

$$1-a^2+a^4-a^6+\ldots+a^{4r} = (1-a^2+a^4-\ldots\pm a^{2r})^2+(a-a^3+\ldots\pm a^{2r-1})^2$$

..... 110

14056. (G. W. Preston, B.A.)—If

$$\frac{x^4}{a}+\frac{y^4}{b}=\frac{1}{a+b} \quad \text{and} \quad x^2+y^2=1,$$

show that

$$\frac{x^{2n+2}}{a^n}+\frac{y^{2n+2}}{b^n}=\frac{1}{(a+b)^n} \ldots\ldots\ldots\ldots 104$$

14058. (Rev. T. Mitcheson, B.A.)—If (h, k) be a point external to the ellipse $a^2y^2+b^2x^2=a^2b^2$, and θ the angle at which a tangent from the point is inclined to the major axis, then

$$2hk = (h^2-a^2)\tan\theta+(k^2-b^2)\cot\theta \ldots\ldots\ldots 74$$

14060. (R. Knowles, B.A.)—Tangents are drawn from a point T (hk), to meet the ellipse $a^2y^2+b^2x^2=a^2b^2$, centre C, in P and Q; prove (1) that

$$CP^2+CQ^2=\frac{2(a^2k^2+b^2h^2-a^2b^2)(a^4k^2+b^4h^2)+2a^4b^4(h^2+k^2)}{(a^2k^2+b^2h^2)^2};$$

(2) if CP, CQ are conjugate, the locus of T is the ellipse

$$a^2y^2+b^2x^2=2a^2b^2 \ldots\ldots\ldots 71$$

14061. (W. C. Stanham, B.A.)—Circles C_1, C_2 intersect at A and B. DE is a chord of C_1, C′E′ a chord of C_2. DE′ meets C_2 again at D′, C′E meets C_1 again at C. If DE and C′E′ are parallel, the anharmonic ratios (ABCD) and (ABC′D′) are equal. Conversely, if (ABCD) and (ABC′D′) are equal, DE and C′E′ are parallel 66

14063. (Robert W. D. Christie.)—The sum of any $\frac{1}{2}(p-1)$ *consecutive odd* squares, *plus* a triangular, may be resolved into the same number of *consecutive even* squares, *plus* a triangular, when the triangulars are non-consecutive, if $p>3$. *Ex. gr.*,

$$3^2+5^2+7^2+1 = 2^2+4^2+6^2+28.$$

[This theorem has no connexion with Dr. Glaisher's similar theorem, viz., that a square *plus* a triangular may be resolved into another square *plus* a triangular] 69

14064. (I. Arnold.)—Through a given point P between the legs of a given angle A, to draw a line BPC so that BP may be m times PC, and then show (without duplicate ratio) that equiangular triangles are as the squares of the homologous sides, using Book I. only ..... 115.

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14066. (Professor Franz Rogel.)—Prove that

$$\binom{6p+3}{6}B_3-\binom{6p+3}{1}B_6+\binom{6p+3}{18}B_9-+\ldots$$
$$\ldots+(-1)^{p+1}\binom{6p+3}{6p}B_{3p}=2p \quad\ldots\ldots(1),$$

$$\binom{6p+5}{2}B_1-\binom{6p+5}{8}B_4+\binom{6p+5}{14}B_7-+\ldots$$
$$\ldots+(-1)^{p}\binom{6p+5}{6p+2}B_{3p+1}=2p+\tfrac{5}{3}\ldots(2),$$

$$\binom{6p+1}{4}B_2-\binom{6p+1}{10}B_5+\binom{6p+1}{16}B_8-+\ldots$$
$$\ldots+(-1)^{p}\binom{6p+1}{6p-2}B_{3p-1}=p+\tfrac{1}{6}\ \ldots(3),$$

where B_1, B_2, ... denote the Bernoullian numbers 34

14067. (Professor E. Lemoine.)—Soit M un point dont les coordonnées normales sont x, y, z par rapport à un triangle de référence ABC; AM, BM, CM coupent BC, CA, AB en A′, B′, C′; A_1, B_1, C_1 sont les milieux des hauteurs du triangle ABC. Trouver le lieu de M si A′A_1, B′B_1, C′C_1 sont concourantes en un point N 76

14069. (Rev. T. Roach, M.A.)—Sum to infinity the series
$\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\frac{5}{14}$ ..... 103.

14070. (Francis Rountree, M.A.)—Solve the following sets of simultaneous equations:—

$$\left.\begin{aligned} x^2-5xy+5y^2+3x-5y+3&=0\\ 3(x^2-2xy+4y^2)+2(x-2y)-1&=0\end{aligned}\right\}\ \ldots\ldots(1),(2),$$

and

$$\left.\begin{aligned}(1)\quad 2x^2+3xy&=22x-18\\ (2)\quad 2y^2-\ xy&=22y-54\end{aligned}\right\}\ \ldots\ldots 108$$

14071. (Professor G. B. Mathews, F.R.S.)—Prove that the envelope of the line $\lambda\xi(\mu\eta^2-\nu\zeta^2)x+\mu\eta(\nu\zeta^2-\lambda\xi^2)y+\nu\zeta(\lambda\xi^2-\mu\eta^2)z=0$,
subject to the condition $\eta\zeta+\zeta\xi+\xi\eta=0$,
is, in general, a curve of the third class and fourth order; and investigate the relations connecting λ, μ, ν in any exceptional cases.

Show how this analytical problem is connected with the following:—
The sides BC, CA, AB of a triangle inscribed in a conic meet a fixed line u in the points α, β, γ; P is any point on the conic, and Pα, Pβ, Pγ meet the conic again in A′, B′, C′. It may be proved that AA′, BB′, CC′ meet in a point Q on u; it is required to find the envelope of the line PQ as P moves round the conic. [A special case is the envelope of the axis of a parabola which touches three given lines]..... 76

14072. (Professor Neuberg).—Résoudre le système
$\sin(y+z+u)=a\sin x$, $\sin(z+u+x)=b\sin y$,
$\sin(u+x+y)=c\sin z$, $\sin(x+y+z)=d\sin u$ 105

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14073. (Prof. S. Sircom, M.A.)—If n is a positive integer, prove that $\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta = \frac{1}{(n-1)!\,2^n}\left\{n^{n-1} - n(n-2)^{n-1} + \frac{n(n-1)}{1.2}(n-4)^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}(n-6)^{n-1} + \ldots\right\}\pi$.

[I have only seen this question worked for low values of n] 115

14075. (Professor A. Droz-Farny.)—Le cercle inscrit du triangle ABC touche les côtés en D, E, F. Soit S le point de contact de ce cercle avec le cercle d'Euler. Démontrez que la droite de Simson de S par rapport au triangle DEF est parallèle à la droite d'Euler de ce triangle. Théorème analogue pour les cercles ex-inscrits? 106

14077. (Professor Sanjána.)—If the arcs joining the mid-points of the sides of a spherical triangle be each 90°, the sum of the squares of the cosines of the half-sides of the triangle shall be unity. (Suggested by Quest. 13661) 100

14080. (D. Biddle.)—Prove that N is factorizable ($N = 6n \pm 1$) when $(N-1)^2 - 576\lambda = \square$ and $(N+1)^2 - 576\lambda = \square$ .. ... (α, β), provided $\lambda > 0$ and integral, of the same value in each equation. [(α) represents the square of the difference of the factors, (β) represents the square of their sum. When N is prime, *both* equations hold good *only when* $\lambda = 0$] 93

14100. (Professor Elliott, F.R.S.)—ABC, A′B′C′ are two triangles inscribed in a conic, and PQR is the Pascal line of the hexagon AB′CA′BC′, P being the intersection of B′C and BC′, Q of C′A and CA′, and R of A′B and AB′. Prove that, if XYZ be an inscribed triangle which is copolar both with ABC and A′B′C′, with vertices homologous in the order named, then XP, YQ, ZR intersect on the conic, and the two poles lie on the line PQR. Prove also that all inscribed triangles XYZ which are copolar both with ABC and A′B′C′ can be constructed by joining P, Q, R to any the same point of the conic, and taking for vertices the second intersections of the conic with the joining lines 73

14104. (Professor Elliott, F.R.S.)—X, Y, Z, O_1, O_2 are five points on any cubic; O_1X, O_1Y, O_1Z meet the cubic again in A, B, C respectively, and O_2X, O_2Y, O_2Z in A′, B′, C′. Prove that the points P, Q, R, where BC′ and B′C, CA′ and C′A, AB′ and A′B respectively intersect, also lie on the cubic; and that PX, QY, RZ meet in a point on the cubic. Also prove conversely that, if O be a tenth point on any cubic of the pencil which passes through six points A, B, C, A′, B′, C′ and through P, Q, R derived from the six as above, and if PO, QO, RO meet the cubic again in X, Y, Z respectively, then AX, BY, CZ meet in a point on the cubic, as also do A′X, B′Y, C′Z 90

14111. (Professor A. Droz-Farny.)—Menons par l'orthocentre H d'un triangle ABC deux transversales Δ et Δ′ perpendiculaires l'une sur l'autre. Il s'agit de démontrer que ces perpendiculaires déterminent sur chaque côté du triangle un segment dont les points milieux α, β, γ sont en ligne droite 89

14120. (Rev. T. Roach, M.A.)—Prove geometrically
$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$ 109

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14121. (C. N. Cama.)—If two parabolas touch at P, and their axes meet at O, the intersections of the focal distance to P of one with the axis of the other lie on a circle which passes through P and O. Hence, if S, S′ be fixed points on fixed lines meeting at right angles in O, and two systems of parabolas be drawn having S, S′ for their foci, and axes OS, OS′, such that each parabola of one system touches the corresponding parabola in the other system, all the points of contact lie on a fixed circle. Generally, four parabolas of one system can be drawn to touch a given parabola of the other system. If the axes be parallel, show that the corresponding locus is a straight line 73

14124. (G. H. Hardy.)—Prove that

$$\int_0^{\frac{1}{4}\pi} \log\tan\phi\,d\phi = \tfrac{1}{3}\int_0^{\frac{1}{4}\pi} \log\tan\phi\,d\phi = -4\int_0^{\frac{1}{4}\pi} \tanh^{-1}\tan\phi\,d\phi \ldots 100$$

14125. (Discipulus.)—If $(1+2x)^m/(1-x)$ be expanded in ascending powers of x, prove that the coefficient of any power of x greater than $m-1$ is 3^m, m being a positive integer 107

14127. (Rev. T. Mitcheson, B.A.)—Show that

$$(p+q)\,!/p\,!\,q\,! = 1+pq+p(p-1)\,q(q-1)/(2\,!)^2 + p(p-1)(p-2)\,q(q-1)(q-2)/(3\,!)^2+\ldots \quad \ldots 107$$

14129. (R. Knowles, B.A.)—Prove that the $(r+1)^{\text{th}}$ terms in the expansions of $(1+x)^{m/n}$ and $(1-x)^{-m/n}$ are respectively

$$\frac{(m+n-nr)\,(m+2n-nr)\,(m+3n-nr)\ldots \text{to } r \text{ terms}}{n^r\,.\,r\,!};$$

$$\frac{(n-m-nr)\,(2n-m-nr)\,(3n-m-nr)\ldots \text{to } r \text{ terms}}{n^r\,.\,r\,!} \ldots\ldots 107$$

14132. (I. Arnold.)—Describe a square in a given sector having two angular points on the arc and the other two on the radii 103

14146. (R. F. Davis, M.A.)—If OP, OP′ are tangents from an external point O to a given circle, and any point T is taken upon the line joining their middle points R, R′ from which tangents are drawn whose chord of contact intersects RR′ in U, prove that OT, OU are at right angles 108

14154. (H. J. Woodall, A.R.C.S.) — Find the value of x in the following table:—

0	·493 107 418 043 066 689
1	·502 687 750 553 717 016
2	·512 251 521 229 044 033
3	·521 798 422 844 633 606
4	·531 328 149 182 931 742
5	·540 840 395 051 310 490
6	·550 334 856 300 060 834
7	x
8	·569 269 213 661 871 302
9	·578 708 506 850 997 002 120

14155. (G. H. Hardy.)—Prove, by contour integration, or otherwise, that

$$\int_0^\infty e^{n\cos\alpha x}\sin(\beta x + n\sin\alpha x)\,(dx)/x = \tfrac{1}{2}\pi e^n.$$

What conditions, if any, must α, β, n satisfy? 117

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14156. (J. Watson.)—In a given circle is a fixed chord equal to the radius, and also a point (not the centre). Describe a second circle that shall touch the given circle and the chord whilst passing through the given point 92

14159. (Rev. J. Cullen.)—Prove (1) that the Pascal line of the hexagon DD'EE'FF', formed by the intersections of the sides of ABC with any "T" circle, is the radical axis of "T" and the circumcircle; (2) the intersections of the homologous sides of DEF and D'E'F' are the vertices of a triangle homothetic with ABC, and the homothetic centre lies on Kiepert's hyperbola 105

14161. (W. C. Stanham, B.A.)—If the tangents to a conic from any point on one of its directrices intersect any fixed straight line at points P and Q, the other two tangents to the conic from P and Q intersect on a fixed straight line..... 101

14613. (I. Arnold.)—Given the length l of an inclined plane, and its height h. A body descends from the top of the plane. Find at what distance from the top of the plane the body will begin to *describe* a space equal to the height in the same time that it would have fallen perpendicularly through the height 91

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MATHEMATICS

FROM

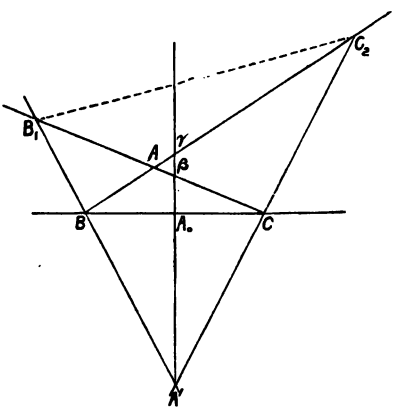
THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

13955. (Professor Neuberg.)—On construit sur les côtés d'un triangle ABC trois triangles isoscèles semblables A'BC, B'CA, C'AB. Les droites BA', BC' coupent AC en B_1, B_2; les droites CB', CA' coupent BA en C_1, C_2; enfin, les droites AC', AB' coupent CB en A_1, A_2. Cela posé, les droites B_1C_2, C_1A_2, A_1B_2 enveloppent trois hyperboles, qui touchent deux des côtés du triangle ABC et ont pour asymptote le troisième côté; ces courbes ont pour tangente commune la droite de Lemoine de ABC.

Solution by Professor A. Droz-Farny.

Soit A_0 le point milieu de BC; le lieu géométrique du sommet A' est la médiatrice correspondant au côté BC; elle rencontre AC et AB respectivement en β et γ.



Les rayons BA' et CA' décrivent deux faisceaux perspectifs, donc les ponctuelles B_1 et C_2 sont homographiques; la droite B_1C_2 enveloppe une conique tangente aux côtés AC et AB.

En faisant coïncider A' avec β ou γ, on constate qu'au point β de la série B_1 ou qu'à γ de la série C_2 correspond chaque fois dans l'autre série le point A; il en résulte que $\beta\gamma$ est la polaire de A. Si A' coïncide avec A_0, B_1 tombe en C et C_2 en B; la conique est donc tangente aux trois côtés du triangle, et, d'après un théorème connu, le conjugué harmonique de A_0 par rapport

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aux points B et C, c'est-à-dire le point infini de BC, sera le point de contact sur cette droite. La conique est donc une hyperbole dont BC est une des asymptotes. Comme il est facile de le voir, la seconde asymptote correspond au point infini de la médiatrice, et les trois centres des hyperboles sont en ligne droite.

En faisant coïncider A′ avec le pôle de BC par rapport au cercle circonscrit ABC, la tangente B_1C_2 devient évidemment la droite de LEMOINE du triangle.

[Mr. G. N. BATES solves the Question as follows:—

The coordinates of A′ are $-\sin\theta$; $\sin(C+\theta)$; $\sin(B+\theta)$;

" B′ " $\sin(C+\theta)$; $-\sin\theta$; $\sin(A+\theta)$;

" C′ " $\sin(B+\theta)$; $\sin(A+\theta)$; $-\sin\theta$.

The equation of B_1C_2 is

$$\alpha\sin(C+\theta)\sin(B+\theta)+\beta\sin\theta\sin(B+\theta)+\gamma\sin\theta\sin(C+\theta)=0.$$

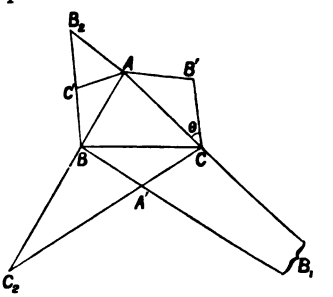
On arranging this equation in powers of $\sin\theta$ and $\cos\theta$, the envelope is seen to be $(a\alpha+b\beta+c\gamma)^2=4bc\alpha(\alpha\cos B\cos C+\beta\cos B+\gamma\cos C)$ ... (i.), an hyperbola having $\alpha=0$ for its asymptote.

Multiplying throughout by b^2c^2, this equation may be written

$$(b\cos C-c\cos B)^2(bc\alpha+ca\beta+ab\gamma)^2+(c^2\beta+b^2\gamma)^2-2(b\cos C-c\cos B)\times(b^2\gamma-c^2\beta)(bc\alpha+ca\beta+ab\gamma)=0;$$

i.e., (i.) touches $\beta=0$, $\gamma=0$, and $bc\alpha+ca\beta+ab\gamma=0$.

Similarly for the envelopes of C_1A_2, A_1B_2, which have one side of ABC as one asymptote, have the other two sides as tangents, and are each touched by $bc\alpha+ca\beta+ab\gamma=0$.]



13889. (G. H. HARDY.)—Sum $1-\frac{2}{5}+\frac{3}{9}-\frac{4}{13}+\ldots$ ad inf.

Solution by ANNIE CHARTRES; Rev. T. ROACH, M.A.; H. W. CURJEL, M.A.; Professor SANJÁNA, M.A.; *and others.*

$$1-\tfrac{2}{5}+\tfrac{3}{9}-\tfrac{4}{13}+\ldots=\int_0^1\frac{dx}{(1+x^4)^2}=\tfrac{1}{8}+\tfrac{3}{4}\int_0^1\frac{1}{1+x^4}\,dx$$
$$=\tfrac{1}{8}+\tfrac{3}{16}\sqrt{2}\left\{\log(\sqrt{2}+1)+\tfrac{1}{2}\pi\right\}.$$

See TODHUNTER'S *Integral Calculus*, p. 36.

[It is uncertain which sign precedes the term at infinity. The respective results, however, differ by $\frac{1}{4}$. The value given in the above solution is the mean.]

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14037. (I. Arnold.)—From a given point within a triangle, to draw three right lines trisecting the triangle.

Solution by G. W. Preston, B.A.

Let ABC be the triangle, and P the given point within it.

Draw PS any line cutting AC in S. Join PA.

Draw SF parallel to PA, and meeting BA produced in F.

Then, clearly,

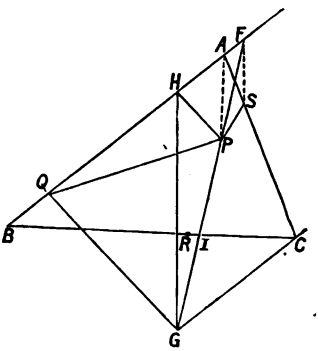
$$\triangle PFA = \triangle PSA.$$

Cut off $FH = \frac{1}{3}AB$.

Through C draw CG parallel to AB, meeting FP produced in G.

Join PH, and draw GQ parallel to PH and meeting AB in Q. Join PQ. Then QPSA is one-third of the triangle. For join GH. Then, clearly,

$$\triangle PHQ = \triangle PHG. \text{ (Euc. I. 37)}$$



To each add PAH; therefore

$$\text{figure } PAQ = \text{figure } PAHG \quad \text{and} \quad \triangle PSA = \triangle PFA.$$

Therefore $$\text{figure } PSAQ = \triangle GHF = \tfrac{1}{3}\triangle ABC,$$

since its base is $\frac{1}{3}AB$ and its altitude is the same.

Similarly, by joining PB and going through a like process, we can find R on BC, such that figure $PQRB = \frac{1}{3}$ of triangle.

The same method would divide the triangle into any number of equal parts by lines drawn from P.

[Lt.-Col. Cunningham solves the Question as follows:—

P the given point within the triangle ABC; G the centroid of ABC. Three cases arise.

(1) If P is *at* G, then PA, PB, PC trisect the triangle.

(2) If P is not at G, draw Pl, Pm, Pn; GL, GM, GN, the three perpendiculars from P and G on the sides. In this case, one (and only one) of Pl, Pm, Pn can be equal to GL, GM, GN, respectively. Suppose

$$Pl = GL, \quad Pm > PM, \quad Pn < PN.$$

Then $$\triangle BPC = \triangle BGC = \tfrac{1}{3}\triangle BAC.$$

Take Cm' on CA so that $Cm' \cdot Pm = CA \cdot GM$;

therefore $$\triangle CPm' = CPA = \tfrac{1}{3}\triangle ABC.$$

Thus Pn', PC, PB are the lines.

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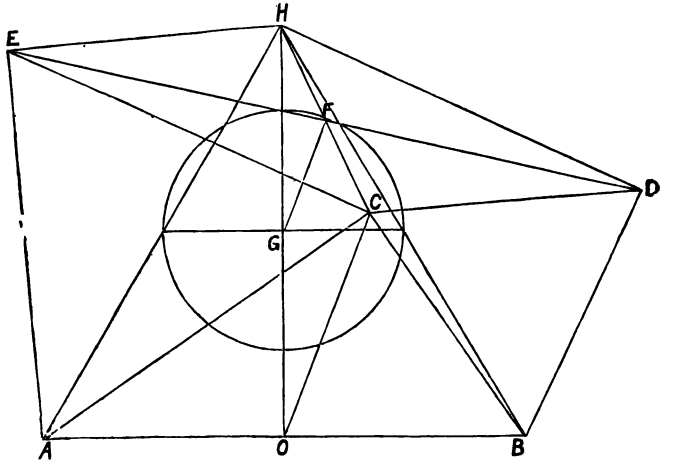
(3) The only other case is one of Pl, Pm, Pn < GL, GM, GN, respectively (suppose Pl < GL); and the other pair of vectors from P each greater than the corresponding vector from G, *i.e.*, Pm > GM, Pn > GN.
Take Am' on AC so that $Am' . Pm = AC . GM$,
and take An' on AB so that $An' . Pn = AB . GN$.
Therefore $\triangle APm' = AGC = \frac{1}{3}ABC$ and $\triangle APm' = AGB = \frac{1}{3}ACB$.
Hence PA, Pm', Pn' are the lines required.]

13886. (JOHN F. WILKINSON, B.A.)—If equilateral triangles be described on the sides of a right-angled triangle whose hypotenuse is given in magnitude, the locus of the middle point of the line joining their vertices is a circle. (By First Book Euclid, CASEY'S *Sequel to Euclid*.)

Solution by R. TUCKER, M.A.; *and* C. JOSS, M.A.

Describe the equilateral triangle ABH on the hypotenuse, and bisect the altitude HO in G, and complete the figure as in diagram. Since the triangles EAH, ABC, BDH are congruent, therefore

$$EH = CB = CD \quad \text{and} \quad HD = AC = CE;$$



hence ECDH is a parallelogram; therefore, if CH be drawn, it passes through F and is bisected at F. Hence

$$FG = \tfrac{1}{2}CO = \tfrac{1}{4}OB = \text{constant};$$

therefore, &c.

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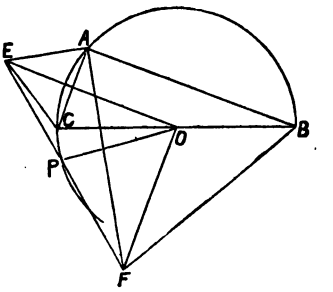
[Mr. CURJEL further observes that the theorem holds good when the equilateral triangles are described on the sides diversely: Let the triangles be described one on the same side and the other on the opposite side to the right-angled triangle.



Then $\Delta FAE = BAC$ in all respects. Also $\angle FOE$ is a right angle, since FO and OE are perpendicular to AB, AC. Hence it may easily be shown by Euclid, Book I., that a circle, centre P, passes through FOAE; therefore

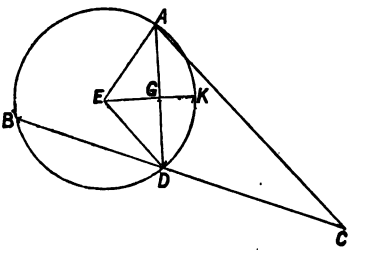
$$OP = \tfrac{1}{2}FE = \tfrac{1}{2}BC = OC;$$

therefore locus of P is circle ABC.]

13935. (J. TIMMER.)—Construct a triangle ABC, having given the side BC, the angle B, and the length of the bisector of A.

Solution by I. ARNOLD.

Let AD be the given bisector, and describe the circle ABD containing the angle B = given angle and E the centre. Take a point B on circumference and join BD, BA. Produce BD, and make angle DAC = DAB. If BC = given base, the thing is done. Take other points B′ and B″, and do the same as in case of B, which will give the points C′ and C″—this can evidently be done till we get a line B′C′ or B″C″ equal to the given base.



Consequently the points C, C′, C″ are on some curve, and the question is solved by inflecting through D a line of given length and terminated at one extremity by the curve, and the other extremity on the circumference of the circumcircle ABD.

The question appears to be one of those which can only be solved conditionally.

[Supposing the above curve to be drawn, the difficulty is scarcely lessened, because the line, of length equal to given base, has yet to be drawn through D, meeting the said curve at one extremity and the circle at the other. The problem seems to resist the ordinary methods of geometry and algebra. But there are some interesting

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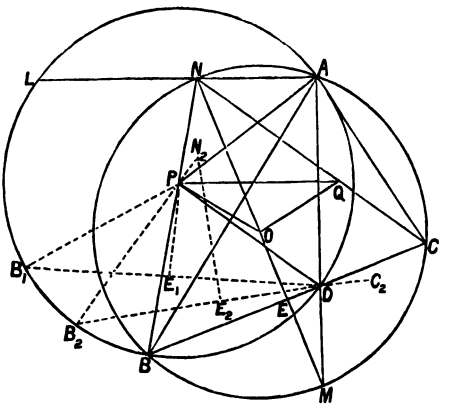
facts to be noticed in connexion with it. Thus, having AD and the angle B given, we can find the relation to AD of P, the circumcentre of ABD; and we know that BP (produced) intersects the perpendicular bisector of BC on AL, which is at right angles with AD; also that N, the point of intersection, is on the circumcircle of ABC, and that CN contains Q, the circumcentre of ADC. Moreover, P and Q are equidistant from O, the circumcentre of ABC, and NE and AD intersect at M on the circumcircle of ABC.



Knowing that BC passes through D, we can allow C_1 to coincide with D (if BC be less than a diameter of ABD) and the other extremity to be at B_1; N_1 will then coincide with P. For B_2, the intersection will be at N_2, and so on, the origin of the curve formed by N_1, N_2, &c., being at P on the line B_1P produced, and the polar equation being $\rho = \frac{1}{2}BC/\sin(\lambda - \frac{1}{2}\theta) - k$, where k = circum-radius of ABD and $\lambda = \angle B_1PE_1$. The intersection of this curve with AL determines N, and the line NPB determines B.]

14033. (R. Knowles, B.A.)—Prove that the $(r+1)$th term in the expansion of $\dfrac{1}{(1-x)^n}$ is $\dfrac{(r+1)(r+2)\ldots(r+n-1)}{n-1!}x^r$.

Solution by F. H. Peachell, B.A.; *and* Rev. T. Mitchesson, B.A.

The $(r+1)$th term of $(1-x)^{-n}$

$$= \frac{n(n+1)\ldots(n+r-1)}{r!}x^r - \frac{n+r-1!}{r.\,n-1!}x^r = \frac{(r+1)(r+2)\ldots(n+r-1)}{n-1!}x^r.$$

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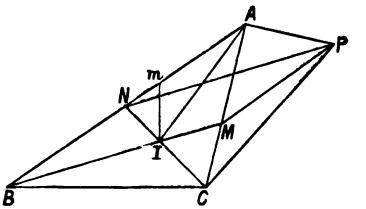
[The PROPOSER adds:—

$$(1-x)^{-n} = 1 + \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 + \&c.,$$

and each term of this is evidently obtained by putting $r = 1, 2, \&c.$, in the formula in the question.]

14030. (H. MACCOLL, B.A.)—Show that in any triangle the bisector of the less of two unequal angles is greater than the bisector of the greater.

Solution by R. F. DAVIS, M.A.; C. JOSS, M.A.; *and many others.*



Let ABC be a triangle; BM, CN the bisectors of the angles at the base. Then, if C > B, it is required to show that BM > CN.

Upon AB take $Am = AM$. Then

$\angle AmI = AMI$

$= C + \frac{1}{2}B > B + \frac{1}{2}C > ANI.$

Therefore AN > AM,

and perp. from N on AC (or BC) > perp. from M on AB (or BC).

Hence $\triangle BNC > BMC$, or $\triangle BNI > CMI$, or BN > CM.

Complete the parallelogram MBNP. Then

$$\angle MCP > MPC \text{ and } \angle NCM > NPM.$$

Hence $$NCP > NPC \text{ and } NP\,(BM) > CN.$$

The above proof only involves the first book of Euclid.

[*Otherwise*:—Let A > B, and let the bisectors AD, BE meet the circumcircle (with centre O) in F, G, the mid-points of the respective arcs. Then $a > b$, $h_b > h_a$.

(1) When $C + \frac{1}{2}A < \frac{1}{2}\pi$; that is, when $ADC > \frac{1}{2}\pi$, AF > BG (chords as well as arcs),

and $$\cos AFO > \cos BGO.$$

But $AD : BE = h_a \sec AFO : h_b \sec BGO = h_a \cos BGO : h_b \cos AFO$;

and both factors representing AD, in the latter ratio, are, as shown above, smaller than those similarly representing BE. Therefore BE > AD.]

(2) When $C + \frac{1}{2}A > \frac{1}{2}\pi$, that is, when $ADC < \frac{1}{2}\pi$, draw DD′, EE′ across the triangle parallel to AB. Then the perpendicular on AB, from any point of the median from C (which bisects DD′, EE′), is nearer A than B, and it follows that BE > AE′ and BD′ > AD. Moreover, CD : DB = CD′ : DA′, and CE : EA > CD : DB. Therefore DD′ is nearer than EE′ to C. But, since $ADC < \frac{1}{2}\pi$, lines drawn from A to DB increase in length as they approach B; therefore AD < AE′ < BE.

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14009 & 14024. (G. Hale Puckle, M.A.)—(14009.) The angle between the positive sides of a pair of intersecting straight lines, represented by $\phi(xy) = 0$, is greater, equal to, or less than a right angle, as $a + b$ is greater, equal to, or less than 0.

(14024.) If $\phi(xy) = 0$ represents a pair of real intersecting straight lines, then (with the conditions of Quest. 13535, No. 437) the equations

$$\alpha(ax + hy + g) + \beta(hx + by + f) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\beta(ax + hy + g) - \alpha(hx + by + f) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2),$$

represent the bisectors of the angles between them; and (1) or (2) is the bisector of the acute angles, as $a + b >$ or < 0.

Solutions by the Proposer.

(14009.) All hyperbolas of which the lines are asymptotes, and which lie in this and the opposite angles, are represented by $\phi(xy) = p$, where p is a positive quantity; and the discriminant of all these is positive. Hence the eccentricity of all, corresponding to real foci, is determined by $e^2 = 2R/(R - a - b)$. [See Quest. 13535, *Educational Times*, No. 437.] The lines are one of these hyperbolas, and their eccentricity is evidently $= \sec\theta$, where 2θ is the angle between the positive sides, since the intersection of the lines is the focus, and the conjugate axis of the hyperbolas the directrix. Hence

$$\cos^2\theta = (R - a - b)/2R, \quad \cos 2\theta = -(a + b)/R, \quad \&c.$$

(14024.) For (2) is the transverse axis of all hyperbolas drawn in the angle between the positive sides of the lines $\phi(xy) = 0$, as asymptotes, &c. [See Quest. 14009.]

14025. (Rev. W. Allen Whitworth, M.A.)—A number is expressed in the scale of notation whose radix is r, and its final digit is prime to r. Show that some power of the number will have its final digit unity. *E.g.*—In the common scale, if a number end in 3 or 7, its fourth power ends in 1; if it end in 9, its square ends in 1.

Solution by H. J. Woodall, A.R.C.S.

This depends upon the fact that with any number m as modulus, and some other number p less than and prime to m, the residues of the powers of p will be a series of numbers in cyclical order. At some step therefore we must obtain the number p. At the previous step therefore we should have had 1.

[Lt.-Col. Allan Cunningham, R.E., puts it thus:—Here

$$N = a_0 + a_1 r + a_2 r^2 + \ldots + a_{n-1} r^{n-1} \equiv a_0 \pmod{r},$$

therefore $$N^n \equiv a_0^n \pmod{r};$$

therefore $$N^{p-1} \equiv a_0^{p-1} \pmod{r} \quad \text{(if } r \text{ is prime)}$$

$$a \equiv {}_0^{\phi(r)} \pmod{r} \quad \text{(when } r \text{ is } \textit{composite}\text{)},$$

where $\phi(r) =$ totient of r.

Mr. Curjel puts it as follows:—The final digit of the power depends on the final digit of the number, but $x^{\phi(r)} \equiv 1 \pmod{r}$, if x is prime to r. Hence the required result.]

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13995. (Rev. W. A. WHITWORTH, M.A.)—Any number N is the sum of either one, two, or three triangular numbers. Show that the three cases may be discriminated as follows:—

(1) If $8N+1$ be a square, N is a triangular number.

(2) If $8N+1$ be not a square, and $4N+1$, when all square factors are cast out of it, contains no prime factor of the form $4n-1$, then N is the sum of *two* triangular numbers.

(3) In all other cases N is the sum of *three* triangular numbers.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

(1) If $N = \frac{1}{2}n(n+1)$, then $8N+1 = 4n^2+4n+1 = \square$, and *vice versa*.

(2) If $N = \frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)$, then

$$4N+1 = 2(m+\tfrac{1}{2})^2+2(n+\tfrac{1}{2})^2 = (m+n+1)^2+(m-n)^2.$$

Therefore every factor of N is of form $(\alpha^2+\beta^2)$, and also of form $(4n+1)$.

Conversely, if $(4N+1)$ contain no prime factor of form $(4n-1)$, it must be of form $(\alpha^2+\beta^2)$, and therefore $= T_m+T_n$, as above.

(3) In all other cases, $N = T_m+T_n+T_r$ (by FERMAT's theorem).

14006. (W. C. STANHAM, B.A.)—Suggested by Quest. 13968.—A conic can be described having its asymptotes parallel to the sides AB, AC of a triangle ABC, passing through B and C, and having a focus at the point of contact with BC of the inscribed circle (or of an escribed circle) of ABC.

Solution by the PROPOSER.

The theorem results by reciprocation from (13968) when the conic in the latter is a parabola.

7476. (D. EDWARDES.)—If $xyz = (2-x)(2-y)(2-z)$, find

$$\int_0^1\int_0^1 xyz\,dx\,dy.$$

Solution by Professor S. SIRCOM, M.A.

For x and y write $1-x$, $1-y$; the limits are the same, and $z = \{(1+x)(1+y)\}/(1+xy)$, the integral becomes

$$\int_0^1\int_0^1 \frac{(1-x^2)(1-y^2)}{1+xy}\,dx\,dy.$$

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Integrating with respect to x, we have

$$\int_0^1 \frac{1-y^2}{y^2}\left\{1-\tfrac{1}{2}y-\frac{1-y^2}{y}\log(1+y)\right\}dy$$
$$=\int_0^1 (1-y^2)\left\{1-\tfrac{1}{2}y-\tfrac{1}{3}(1-y^2)+\tfrac{1}{4}y(1-y^2)\ldots\right\}dy,$$

which reduces to $\frac{1}{6}\pi^2-\frac{5}{4}$.

[For another solution of this Question, see vol. XLI., p. 116.]

14066. (Professor FRANZ ROGEL.)—Prove that

$$\binom{6p+3}{6}B_3-\binom{6p+3}{12}B_6+\binom{6p+3}{18}B_9-+\ldots$$
$$\ldots+(-1)^{p+1}\binom{6p+3}{6p}B_{3p}=2p \quad\ldots\ldots(1),$$

$$\binom{6p+5}{2}B_1-\binom{6p+5}{8}B_4+\binom{6p+5}{14}B_7-+\ldots$$
$$\ldots+(-1)^{p}\binom{6p+5}{6p+2}B_{3p+1}=2p+\tfrac{4}{3}\ldots(2),$$

$$\binom{6p+1}{4}B_2-\binom{6p+1}{10}B_5+\binom{6p+1}{16}B_8-+\ldots$$
$$\ldots+(-1)^{p}\binom{6p+1}{6p-2}B_{3p-1}=p+\tfrac{1}{6}\ \ldots(3),$$

where B_1, B_2, ... denote the Bernoullian numbers.

Solution by the PROPOSER.

From $$B(\tfrac{1}{2}-x,m)=(-1)^m B(\tfrac{1}{2}+x,m)\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where B denotes the *Bernoullian Function* of the mth order of x, we deduce, supposing m odd,

$$B(-\tfrac{1}{2}-x,m)=-B(\tfrac{1}{2}+x,m),\quad B(-\tfrac{1}{2}+x,m)=-B(\tfrac{1}{2}-x,m)\ \ldots\ (2),$$

wherein

$$\left.\begin{aligned}B(\tfrac{3}{2}+x,m)&=B\{(x+\tfrac{1}{2})+1,m\}=B(x+\tfrac{1}{2},m)+m(x+\tfrac{1}{2})^{m-1}\\ B(\tfrac{3}{2}-x,m)&=B\{(\tfrac{1}{2}-x)+1,m\}=B(-x+\tfrac{1}{2},m)+m(-x+\tfrac{1}{2})^{m-1}\end{aligned}\right\}(3).$$

Therefore, by addition,

$$B(-\tfrac{1}{2}-x,m)+B(-\tfrac{1}{2}+x,m)$$
$$=-\left\{B(\tfrac{1}{2}+x,m)+B(\tfrac{1}{2}-x,m)+m(\tfrac{1}{2}+x)^{m-1}+m(\tfrac{1}{2}-x)^{m-1}\right\};$$

or, since the first two terms of the right hand vanish, according to the equation (1), m odd,

$$B(-\tfrac{1}{2}-x,m)+B(-\tfrac{1}{2}+x,m)=m(\tfrac{1}{2}+x)^{m-1}+m(\tfrac{1}{2}-x)^{m-1}\ldots\ldots\ (4);$$

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hence, for $x = \frac{1}{2}u$,

$$\mathrm{B}\left\{\tfrac{1}{2}(-1+u), m\right\} + \mathrm{B}\left\{\tfrac{1}{2}(-1-u), m\right\}$$
$$= -(m/2^{m-2})\,\tfrac{1}{2}\left\{(1+u)^{m-1} + (1-u)^{m-1}\right\}$$
$$= -2^{-m+2}\, m\,(1-u^2)^{\frac{1}{2}(m-1)} \cos\left\{(m-1)\arctan iu\right\} \quad \ldots\ldots (5),$$
$$i = \sqrt{(-1)}.$$

Putting $u = i\sqrt{3}$, we obtain

$$\mathrm{B}\left\{\tfrac{1}{2}(-1+i\sqrt{3}), m\right\} + \mathrm{B}\left\{\tfrac{1}{2}(-1-i\sqrt{3}), m\right\} = -2m\cos(m-1)\tfrac{1}{3}\pi \ \ldots (6);$$

or, observing $\mathrm{B}(1, m) = 0, \qquad m > 1,$

$$\tfrac{1}{3}\left\{\mathrm{B}(e^{\frac{1}{3}\pi i}, m) + \mathrm{B}(e^{\frac{3}{3}\pi i}, m) + \mathrm{B}(e^{\frac{5}{3}\pi i}, m)\right\} = -\tfrac{2}{3}m\cos(m-1)\tfrac{1}{3}\pi,$$

where the expression of the left hand may be brought by means of

$$\mathrm{B}(y, m) = y^m - \tfrac{1}{2}my^{m-1} + \binom{m}{2} B_1 y^{m-2} - \binom{m}{4} B_2 y^{m-4} \ldots$$
$$\ldots + (-1)^{\frac{1}{2}(m+1)} \binom{m}{m-1} B_{\frac{1}{2}(m-1)}\, y, \ m \text{ odd},$$

(B_r Bernoullian numbers) in a real form, if $\mu < 6$ in the congruence

$$m \equiv \mu, \mod 6,$$

is known.

Now, separating the three possible cases, we have

(α) $m = 6p+3, \qquad \cos(m-1)\tfrac{1}{3}\pi = \cos\tfrac{2}{3}\pi = -\tfrac{1}{2}$;

therefore $\binom{6p+3}{6} B_3 - \binom{6p+3}{12} B_6 + \binom{6p+3}{18} B_9 - + \ldots$
$$\ldots + (-1)^{p+1} \binom{6p+3}{6p} B_{3p} = 2p \ \ldots\ldots (7).$$

(β) $m = 6p+5, \qquad \cos(m-1)\tfrac{1}{3}\pi = \cos\tfrac{4}{3}\pi = -\tfrac{1}{2}$;

therefore $\binom{6p+5}{2} B_1 - \binom{6p+5}{8} B_4 + \binom{6p+5}{14} B_7 - + \ldots$
$$\ldots + (-1)^p \binom{6p+5}{6p+2} B_{3p+1} = 2p + \tfrac{4}{3} \ldots (8).$$

(γ) $m = 6p+1, \qquad \cos(m-1)\tfrac{1}{3}\pi = \cos 2p\pi = +1$;

therefore $\binom{6p+1}{4} B_2 - \binom{6p+1}{10} B_5 + \binom{6p+1}{16} B_8 - + \ldots$
$$\ldots + (-1)^p \binom{6p+1}{6p-2} B_{3p-1} = p + \tfrac{1}{6} \ldots (9).$$

From these three relations (7), (8), (9), follow equations (1), (2), (3) in the Question.

13921. (Rev. J. Cullen, B.A.)—Taking $\cos(\mathrm{A}-\theta)$, $\cos(\mathrm{B}-\theta)$, $\cos(\mathrm{C}-\theta)$ as the coordinates and R′ the radius of a Tucker-circle, prove that (1) $\mathrm{R}' = 2\Delta/\Sigma a \cos(\mathrm{A}-\theta)$; (2) the circle intersects AB, AC in F and E′ (FE′ being antiparallel to BC), denoting the circle AFE′ by $\mathrm{T_A}$ and the circles BDF′, CED′ similarly obtained by $\mathrm{T_B}$, $\mathrm{T_C}$; then the

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centres of T_A, T_B, T_C lie on the perpendiculars from the vertices on the sides of ABC; (3) the radical axes of these three circles pass through fixed points; (4) their radical centre is the point

$$a \sin (A + \theta)/\sin 2A = \ldots = \ldots,$$

so that it lies on the inverse of the orthocentroidal axis, *i.e.*, a rectangular hyperbola circumscribing ABC.

Solution by Professor K. J. Sanjána, M.A.

If $A\Omega F = B\Omega D = C\Omega E = \theta$, the circle round DEF is a Tucker-circle; its radius being R′, the coordinates of the centre are $\alpha = R' \cos (A - \theta)$,

$\beta = R' \cos (B - \theta)$,

$\gamma = R' \cos (C - \theta)$.

Hence $2\Delta = a\alpha + b\beta + c\gamma$

$= \Sigma R' a \cos (A - \theta)$;

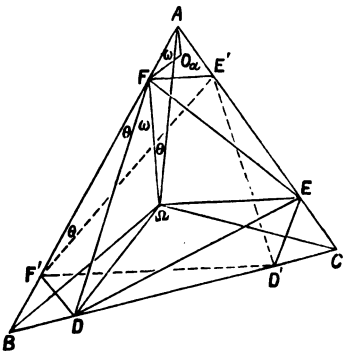
so that

$R' = 2\Delta/\Sigma a \cos (A - \theta)$ ...(1).

Also $R' : R = \Omega F : \Omega A$

$= \sin \omega : \sin (\theta + \omega) = k : 1$,

suppose. The triangle AFE′ is inversely similar to ABC, so that $O_\alpha AF$ is similar to OAC.



Hence $\cos FAO_\alpha = \frac{1}{2}AF/AO_\alpha = \frac{1}{2}AC/AO = \sin B = \cos (90° - B)$;

thus AO_α is perpendicular to BC. So for the other centres ..... (2).

The circle T_A goes through $A(p_1, 0, 0)$, $F\left[kc \sin (B + \theta),\ kb \sin \theta,\ 0\right]$ and $E'\left[kb \sin (C + \theta),\ 0,\ kc \sin \theta\right]$; hence its equation is

$$\begin{vmatrix} a\beta\gamma + b\gamma\alpha + c\alpha\beta & \alpha & \beta & \gamma \\ 0 & p_1 & 0 & 0 \\ k^2bc^2 \sin (B + \theta) \sin \theta & kc \sin (B + \theta) & kb \sin \theta & 0 \\ k^2b^2c \sin (C + \theta) \sin \theta & kb \sin (C + \theta) & 0 & kc \sin \theta \end{vmatrix} = 0.$$

See Casey, Ed. 2, *Seq.* (329). Reducing, we get

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - \beta kc^2 \sin (B + \theta) - \gamma kb^2 \sin (C + \theta) = 0.$$

So also the equations of T_B and T_C may be written down.

The radical axis of the first two is, therefore,

$$\gamma a^2 \sin (C + \theta) + ac^2 \sin (A + \theta) - \beta c^2 \sin (B + \theta) - \gamma b^2 \sin (C + \theta) = 0,$$

that is, $\cos \theta (\gamma a^2 \sin C + ac^2 \sin A - \beta c^2 \sin B - \gamma b^2 \sin C)$

$+ \sin \theta (\gamma a^2 \cos C + ac^2 \cos A - \beta c^2 \cos B - \gamma b^2 \cos C) = 0.$

Hence, whatever be the value of θ, it passes through the point determined

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by equating to zero the coefficients of $\cos\theta$ and $\sin\theta$. This point will be found to be

$$\frac{\alpha}{\sin(B-C)} = \frac{\beta}{\sin(A-C)} = \frac{\gamma}{\sin C}.$$

So also we may prove for the other two radical axes ..... (3).

The radical axes of T_A, T_B, Tc intersect where

$$\beta c^2 \sin(B+\theta) + \gamma b^2 \sin(C+\theta) = \gamma a^2 \sin(C+\theta) + \alpha c^2 \sin(A+\theta)$$
$$= \alpha b^2 \sin(A+\theta) + \beta a^2 \sin(B+\theta).$$

These equations are satisfied by the point $m \sin 2A / \sin(A+\theta)$, $m \sin 2B/\sin(B+\theta)$, $m \sin 2C/\sin(C+\theta)$, which therefore is the radical centre ..... (4).

The radical centre obviously lies on the curve

$$\frac{\sin 2A \sin(B-C)}{\alpha} + \frac{\sin 2B \sin(C-A)}{\beta} + \frac{\sin 2C \sin(A-B)}{\gamma} = 0,$$

as $\Sigma \sin(B-C)\sin(A+\theta) = 0$. This curve is an equilateral hyperbola circumscribed to ABC, and is the isogonal transformation of

$$\Sigma a \sin 2A \sin(B-C) = 0,$$

which is EULER'S line joining the orthocentre and the centroid.

13875. (B. N. CAMA, M.A.)—P is any point on an ellipse, foci S and H. Obtain the differential relation

$$\frac{d \log SP}{d\theta} = \frac{d \log HP}{d\theta'},$$

where $\theta = \angle PSH$, and $\theta' = \angle PHS$.

Hence or otherwise deduce that

$$\frac{1}{SP}\left\{e^2 . SP^4 - l^2\left(\frac{dSP}{d\theta}\right)^2\right\}^{\frac{1}{2}} + \frac{1}{HP}\left\{e^2 . HP^4 - l^2\left(\frac{dHP}{d\theta'}\right)^2\right\}^{\frac{1}{2}}$$

is constant for all positions of P, $2l$ being the length of the latus rectum, and e the eccentricity. Is any geometrical solution possible?

Solution by LIONEL E. REAY, B.A.; Professor UMES CHANDRA GHOSH; *and others.*

In an ellipse, the angles made by a tangent with the focal distances of its point of contact are equal; hence

$$1/r\, dr/d\theta = 1/r'\, dr'/d\theta',$$

where $r = SP$ and $r' = HP$;

therefore

$$\frac{d \log r}{d\theta} = \frac{d \log r'}{d\theta'},$$

$$\frac{d \log SP}{d\theta} = \frac{e \sin\theta}{1+e\cos\theta}, \qquad \frac{d \log HP}{d\theta'} = \frac{-e \sin\theta'}{1-e\cos\theta'};$$

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therefore $$\frac{e\sin\theta}{1+e\cos\theta} = \frac{-e\sin\theta'}{1-e\cos\theta'},$$

and $$\frac{d\mathrm{SP}}{d\theta} = \frac{\mathrm{C}\sin\theta}{(1+e\cos\theta)^2} \text{ and } \frac{d\mathrm{HP}}{d\theta'} = \frac{-\mathrm{C}\sin\theta'}{(1-e\cos\theta')^2};$$

therefore given expression $= \mathrm{C}\left(\frac{\cos\theta}{1+e\cos\theta} + \frac{\cos\theta'}{1-e\cos\theta'}\right) = \mathrm{K}$ say.

Also $$dr = -dr'; \quad \text{therefore } d\theta'/d\theta = -r/r';$$

$$\frac{d\mathrm{K}}{d\theta} = \mathrm{C}\left(\frac{\sin\theta}{(1+e\cos\theta)^2} - \frac{\sin\theta'}{(1-e\cos\theta')^2}\frac{d\theta'}{d\theta}\right)$$

$$= \mathrm{C}\left(\frac{\sin\theta}{(1+e\cos\theta)^2} + \frac{\sin\theta'}{(1-e\cos\theta')(1+e\cos\theta)}\right)$$

$$= \frac{\mathrm{C}}{1+e\cos\theta}\left(\frac{\sin\theta}{1+e\cos\theta} + \frac{\sin\theta'}{1-e\cos\theta'}\right) = 0;$$

therefore K is constant for all positions of P.

13904. (Professor Umes Chandra Ghosh.)—If ρ, ρ_n be the radii of curvature of a curve and its nth positive pedal at corresponding points whose radii vectores are r, r_n, show that $\frac{r_{n-1}}{\rho_n} + \frac{\rho_{n-1}}{r_{n-2}} = 2$. Hence show that $\frac{\rho_n}{r_{n-1}} = \frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\ldots - \frac{r_1\rho}{r^2}$ where $\frac{r_1\rho}{r^2}$ is the $(n+1)^{\text{th}}$ complete quotient.

Solution by B. H. Steede, M.A.; H. W. Curjel, M.A.; *and* Rev. J. Cullen, B.A.

We have $$\rho_n = \frac{r_n dr_n}{dr_{n+1}}, \text{ \&c., also } r_{n+1} = \frac{r_n^2}{r_{n-1}}.$$

Therefore $$r_{n-1}^2 \frac{dr_{n+1}}{dr_n} = 2r_{n-1}r_n - r_n^2 \frac{dr_{n-1}}{dr_n},$$

or $$\frac{r_{n-1}}{\rho_n} + \frac{r_n}{r_{n-1}^2}\rho_{n-1} = 2 \quad \text{or} \quad \frac{r_{n-1}}{\rho_n} + \frac{\rho_{n-1}}{r_{n-2}} = 2,$$

and, since $$\frac{\rho_n}{r_{n-1}} = \frac{1}{2-(r_n/r_{n-1}^2)\,\rho_{n-1}},$$

second result follows.

13898. (Professor Neuberg.)—Soient (A,B,C), (A′,B′,C′), (A″,B″,C″), (A‴, B‴, C‴) les angles de quatre triangles ABC, ABC′, ABC″, ABC‴ construits sur la même base AB. Démontrer que la condition nécessaire

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pour que les sommets C, C′, C″, C‴ appartiennent à une même circonférence est

$$\begin{vmatrix} \cot A & \cot B & \cot C & 1 \\ \cot A' & \cot B' & \cot C' & 1 \\ \cot A'' & \cot B'' & \cot C'' & 1 \\ \cot A''' & \cot B''' & \cot C''' & 1 \end{vmatrix} = 0.$$

Solution by Rev. J. Cullen, B.A.; *and* Professor Sanjána, M.A.

Taking AB for the axis of x, B for the origin, and C the point (x, y), we have $y = -x \tan B$, $y = (x + c) \tan A$;
therefore $x = -c \tan A/(\tan A + \tan B)$, $y = c \tan A \tan B/(\tan A + \tan B)$.
Substituting these and similar values in terms of A′, A″, A‴, &c., in

$$x^2 + y^2 + 2gx + 2fy + d = 0,$$

we have

$$\begin{vmatrix} \dfrac{\tan^2 A\,(1 + \tan^2 B)}{\tan A + \tan B}, & \tan A, & \tan A \tan B, & \tan A + \tan B \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

on eliminating g, f and d. Subtracting the second column from the first and fourth, we get

$$\begin{vmatrix} \cot A, & \cot B, & \cot C, & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

on dividing the rows by tan A tan B, &c.

13887. (J. J. Barniville, B.A.)—Prove that

$$\frac{19}{1^3 \cdot 2^3} + \frac{127}{3^3 \cdot 4^3} + \frac{331}{5^3 \cdot 6^3} + \frac{631}{7^3 \cdot 8^3} + \ldots = \frac{\pi^2}{4}.$$

Solution by Prof. Sanjána, M.A.; H. W. Curjel, M.A.; *and others.*

The general term

$$= \frac{7 - 36n + 48n^2}{(2n-1)^3 (2n)^3} = -3\left(\frac{1}{4n^2}\right) - 7\left(\frac{1}{8n^3}\right) + \frac{3}{(2n-1)^2} + \frac{1}{(2n-1)^3},$$

by partial fractions. Hence the sum

$$= -\tfrac{3}{4}\Sigma\frac{1}{n^2} + 3\Sigma\frac{1}{(2n-1)^2} - \tfrac{7}{8}\Sigma\frac{1}{n^3} + \Sigma\frac{1}{(2n-1)^3} = -\tfrac{3}{4} \cdot \tfrac{1}{6}\pi^2 + 3 \cdot \tfrac{1}{8}\pi^2 = \tfrac{1}{4}\pi^2,$$

as

$$\Sigma\frac{1}{(2n-1)^3} = \Sigma\frac{1}{n^3} - \tfrac{1}{8}\Sigma\frac{1}{n^3}.$$

See Hogg's Key to Todhunter's *Plane Trigonometry*, p. 318.

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13866. (Professor NEUBERG.) — Un paraboloïde hyperbolique est déterminé par quatre points A, B, C, D et par deux plans P, Q auxquels les génératrices des deux modes sont respectivement parallèles. Construire les génératrices qui passent par A, B, C, D. Trouver l'équation du paraboloïde.

Solution by Prof. E. J. NANSON.

Let planes through D parallel to the directing planes cut the plane ABC in the lines d_1, d_2. The section of the surface by the plane ABC passes through the points A, B, C and through the points at infinity on d_1, d_2. The intersections D_1, D_2 of this section with the lines d_1, d_2 can therefore be found by PASCAL'S theorem. DD_1, DD_2 are then the generators through D.

Taking A, B, C, D as points of reference, let the directing planes, the plane at infinity, and an arbitrary plane be

$$u_r \equiv \alpha_r x + \beta_r y + \gamma_r z + \delta_r w = 0 \quad (r = 1, 2, 3, 4);$$

then the surface is of the form

$$u_1 u_2 = u_3 u_4,$$

and this will pass through the points of reference if

$$\lambda_1 \lambda_2 = \lambda_3 \lambda_4 \quad (\lambda = \alpha, \beta, \gamma, \delta).$$

Thus the required equation is

$$(a_1 x + \ldots)(a_2 x + \ldots) = (a_3 x + \ldots)\{(a_1 a_2/a_3)\, x + \ldots\}$$

or

$$\Sigma\, (xy/\alpha_3\beta_3)\, (\alpha_1\beta_3 - \alpha_3\beta_1)(\alpha_2\beta_3 - \alpha_3\beta_2) = 0.$$

13374 & 13393. (R. CHARTRES.)—If two points P and Q be taken at random in the sides AB, AC of a triangle, find (1) the mean value of the area of the circumcircle of PAQ, and (2) the nth power of the triangle PQR if R be a random point in BC.

Solution by Rev. W. A. WHITWORTH, M.A.

(1) The area of the circle is $\pi PQ^2 \div 4 \sin^2 A$. But

$$E(PQ^2) = E(AP^2 + AQ^2 - 2AP \cdot AQ \cos A)$$
$$= \tfrac{1}{3}b^2 + \tfrac{1}{3}c^2 - \tfrac{1}{3}bc \cos A = \tfrac{1}{12}(b^2 + c^2 + 3a^2)$$

Hence the expectation of the area is

$$\pi \operatorname{cosec}^2 A\, (b^2 + c^2 + 3a^2) \div 48 = (b^2 + c^2 + 3a^2)/12a^2$$

of circumcircle of ABC.

(2) The areas APQ, BPR, CQR, PQR are of equal expectation, subject to the condition that their sum = ABC. Therefore, the expectation of the nth power of any one of them is

$$\Delta^n \div R_n{}^4 = 6\Delta^n/(n+1)(n+2)(n+3),$$

Δ denoting the area of the triangle ABC.

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13994. (R. F. Davis, M.A.)—If ABC is an isosceles triangle, D the middle point of the base BC, and upon BA, CA points P, Q are taken such that $BP \cdot CQ = BD^2$, prove that the envelope of PQ is the circle whose centre is D and which touches AB, AC.

Solution by Professor A. Droz-Farny *and* G. W. Preston, B.A.

De l'égalité $BP \cdot CQ = (BD)^2$ on déduit

$$BP : BD = CD : CQ,$$

et, comme les angles B et C sont égaux, les triangles BDP et CQD sont semblables et par conséquent

$$\angle PDQ = B = C.$$

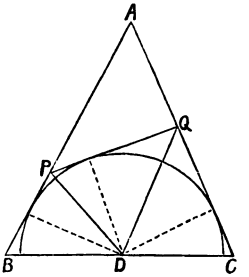
Comme

$$DP : DQ = BD : CQ = CD : CQ,$$

les triangles PDQ et CDQ sont aussi semblables; d'où

$$\angle PQD = DQC,$$

et de même $\angle QPD = DPB$.



Les perpendiculaires abaissées de D sur les côtés CQ, PQ, BP sont donc égales; d'où le théorème.

14032. (Rev. T. Mitcheson, B.A.) — P and Q are the ends of conjugate semi-diameters of an ellipse, whose centre is C and foci H and S. The tangents meet in R, and M and N are the mid-points of HP and SQ: while CR meets PQ in T, MN in L. Show that the loci of M and N are ellipses whose common chord is equal to half the latus rectum of the original ellipse, the chord being bisected in C. The loci of L, T, and R are ellipses.

Solution by F. H. Peachell, B.A.; R. Knowles, B.A.; *and the* Proposer.

Let P be the point ϕ; then Q is point $\phi + \frac{1}{2}\pi$.

(1) Coordinates of R are

$$a(\cos\phi - \sin\phi), \quad b(\cos\phi + \sin\phi);$$

therefore $x^2/a^2 = 1 - 2\sin\phi\cos\phi$,

$$y^2/b^2 = 1 + 2\sin\phi\cos\phi.$$

and locus of R is the ellipse

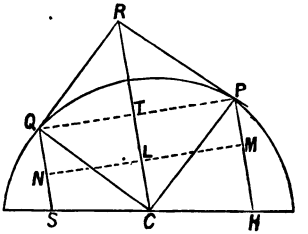
$$x^2/a^2 + y^2/b^2 = 2.$$

(2) Coordinates of M are

$$\tfrac{1}{2}(a\cos\phi + ae), \quad \tfrac{1}{2}b\sin\phi;$$

therefore locus of M is ellipse

$$\left(\frac{2x - ae}{a}\right)^2 + \left(\frac{2y}{b}\right)^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$



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(3) Similarly locus of N is

$$\left(\frac{2x+ae}{a}\right)^2+\left(\frac{2y}{b}\right)^2=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

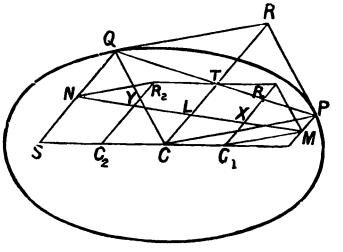
Evidently (α) and (β) intersect on line $x=0$, and (α) intersects $x=0$, where $2y/b=\pm\sqrt{(1-e^2)}$, or $y=\pm b^2/2a$;

therefore common chord $= b^2/a =$ half latus rectum.

(4) CQRP is a parallelogram, and therefore CT = TR; therefore coordinates of T are $\frac{1}{2}a(\cos\phi-\sin\phi)$, $\frac{1}{2}b(\cos\phi+\sin\phi)$, and locus of T is ellipse $x^2/a^2+y^2/b^2=\frac{1}{2}$.

(5) Coordinates of L are $\frac{1}{4}a(\cos\phi-\sin\phi)$, $\frac{1}{4}b(\cos\phi+\sin\phi)$, and locus of L is ellipse $x^2/a^2+y^2/b^2=\frac{1}{8}$.

[Mr. Curjel solves the question as follows:—The loci of M and N are obviously homothetic ellipses foci H, C and S, C. Hence their common chord is their latus rectum through C, and therefore $=\frac{1}{2}$ latus rectum of the original ellipse and is bisected in C.



Let C_1, C_2 be the centres of these ellipses, and let N′ and M′ be the ends of the diameters conjugate to C_1M and C_2N, and R_1, R_2 the intersections of tangents at M, N′, and M′, N. Now, by projecting the ellipse orthogonally into a circle, it is clear that the loci of T and R are ellipses homothetic and concentric with the original ellipse, their axes being respectively $1/\sqrt{2}$ and $\sqrt{2}$ times those of the original ellipse. Let C_1R_1, C_2R_2 cut MN in X and Y. Now CT, C_1R_1, C_2R_2 are equal and parallel, and XY is bisected in L. But triangles NR_2Y, MC_1X are equal in all respects; therefore L is the middle point of CT; therefore locus of L is an ellipse homothetic and concentric with the original ellipse, its axes being $1/(2\sqrt{2})$ of the axes of the original ellipse.]

8842, 8500, 8884, & 12494. (D. Edwardes.)—(8842.) Prove that

$$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k\sin^2\phi)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=4\log 2-4.$$

(8500.) Prove that

$$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=8\log 2-8.$$

(8884, 12494.) Prove that

$$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\{\log(1-k)\}^2}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=16-\tfrac{2}{3}\pi^2+32(\log_e\tfrac{1}{2}e)^2.$$

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Solutions by Professor S. SIRCOM, M.A.

(8842.) Put $k = \sin^2\theta$; then, by the method of Quest. 8465 (Vol. XLVI., p. 75), put $\cos\theta = \cos\mu \sin\omega$, $\cot\phi = \sin\mu \tan\omega$, and, the limits being $\frac{1}{2}\pi$ and 0 throughout, the integral becomes

$$4\int\int \cos\mu \sin\omega \log\sin\omega \, d\omega \, d\mu = 4\int \sin\omega \log\sin\omega \, d\omega = 4\log 2 - 4.$$

(8500.) Similarly, this integral becomes

$$4\int\int \sin\omega \sin\mu \log(\sin\omega\cos\mu)\, d\omega \, d\mu = 8\int \sin\omega \log\sin\omega \, d\omega = 8\log 2 - 8.$$

(8884.) This becomes

$$8\int\int \sin\omega\cos\mu \left\{\log(\sin\omega\cos\mu)\right\}^2 d\omega\, d\mu$$

$$= 16\int \sin\omega (\log\sin\omega)^2 d\omega + 16\left(\int \sin\omega \log\sin\omega\, d\omega\right)^2$$

$$= 16\left\{1 - \tfrac{1}{12}\pi^2 + (\log 2 - 1)^2\right\} + 16(\log 2 - 1)^2 = \text{given result.}$$

[For other solutions of these Questions, see Vol. XLVI., p. 75, and Vol. LVII., p. 77.]

14035. (H. J. WOODALL, A.R.C.S.)—Is anything known about the theory of the following?—When is

$$a^p - 1 \equiv 0 \pmod{q^2},$$

it being given that a is not a power, and neither $a-1$ nor p divisible by q? *E.g.*,

$$(50{,}000)^5 - 1 \equiv 0 \pmod{11^2}.$$

Solutions (1) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(2) *by* C. E. BICKMORE, M.A.

(1) Much of the theory of this congruence has been already worked out; see MATHEWS's *Theory of Numbers*, 1892, Part I., Art. 25–29, &c.

Rewriting it $a^n - 1 \equiv 0$, or $a^n \equiv 1 \pmod{q^2}$,

the following is a brief *résumé*:—There are three quantities a, n, q involved, and a prime to q; two of these are supposed given, to find the *third*. Hence three problems arise.

Problem I.—Given a, q, to find n. First find ξ the *least* exponent satisfying $a^\xi \equiv 1 \pmod{q}$; then $a^{q\xi} \equiv 1 \pmod{q^2}$, and $n = q\xi$ is one value of n, sometimes the least value. In particular, if q be *prime*, then $n = q\xi$ is usually the *least* value of n; if there be a less value, it is $n = \xi$. [*Ex.* $3^5 \equiv 1 \pmod{11}$, and $3^5 \equiv 1 \pmod{11^2}$]. But, if $q \equiv$ product of *primes* $= q_1 q_2 q_3 \ldots$, then $n = q\xi$, or *some* submultiple thereof (depending on the relation of a to q).

Problem II.—Given a, n to find q. A far more difficult problem, involving, in fact, the whole question of factorization. First find *any*

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factor q of (a^n-1), and then find the least exponent ξ which gives $a^\xi \equiv 1 \pmod q$. If $n = q\xi$, or if $n = mq\xi$, then $a^n \equiv 1 \pmod{q^2}$, as required. But, if $n \neq q\xi$, or $n \neq mq\xi$, then the question whether $a^n \equiv 1 \pmod{q^2}$ must be tested specially. In any case, if $a^n \equiv 1 \pmod{q^2}$, then q must be *one* of the quantities satisfying $a^n \equiv 1 \pmod q$.

Problem III.—Given n, q, to find a. Note first that the *number* of distinct (incongruous) roots a each $< q^2$, each satisfying $a^n \equiv 1 \pmod{q^2}$ is $\phi(n)$ the totient of n. Next, if a be any *one* of such roots ($< q^2$), then the residues of the powers of a (*i.e.*, of a^2, a^3, &c.) to modulus q^2 gives the complete set of $\phi(n)$ incongruous roots, each $< q^2$. If these be called a_1, a_2, &c., then all quantities of type $(m_1q^2+a_1)$, $(m_2q^2+a_2)$, &c., are also roots of the congruence. Hence, if *any* one of the roots a_1, a_2, &c., $< q^2$ can be found, all the rest can be found. To find such a root, say a_1, first find a root, say α, of the congruence $a^n \equiv 1 \pmod q$: any root will do. Then a_1 must be of form $mq+\alpha$, and may be shown to be given by solution of the linear congruence in m (as the unknown)

$$n\alpha^{n-1} \cdot m + (\alpha^n - 1) \div q \equiv 0 \pmod q,$$

which is *always possible*.

Ex.—Given $3^5 = 243 \equiv 1 \pmod{11^2}$. This gives 3, 9, 27, 81 as the *complete* set of four (incongruous) roots, each < 121, of the congruence $a^5 \equiv 1 \pmod{121}$. And, since $50{,}000 \equiv 27 \pmod{121}$, therefore

$$50000^3 \equiv 1 \pmod{121}.$$

Ex.—Given $2^5 = 32 \equiv \pmod{31}$. Then a root (a_1) of $a_1^5 \equiv 1 \pmod{31^2}$ will be given by solving $5 \cdot 2^4 \cdot m + (2^5-1) \div 31 \equiv 1 \pmod{31}$, which yields $m = 12$, whence $a_1 = 12 \cdot 31 + 2 = 374$. Hence 374, and the residues of 374^2, 374^3, 374^4 to modulus 961, *i.e.*, 374, 388, 531, 628, is the *complete* set of four incongruous roots, each < 961, of the congruence $a^5 \equiv 1 \pmod{31^2}$.

All the methods above can be extended to solving $a^n \equiv 1 \pmod{q^3, q^4}$, &c.).

(2) If q be a prime > 2, $a^{q-1} = 1 + Mq$ by FERMAT's theorem; therefore

$$a^{q(q-1)} = (1+Mq)^q = 1 + Nq^2,$$

by the binomial theorem. Therefore, if q be a prime, and $p = q(q-1)$,

$$a^p - 1 \equiv 1 \pmod{q^2}.$$

For certain values of a, p may be a submultiple of $q(q-1)$. Thus, if g be any primitive root of q, and

$$g^q \equiv h \pmod{q^2}, \quad g^{q(q-1)} \equiv h^{q-1} \equiv 1 \pmod{q^2};$$

similarly, if $g^{2q} \equiv h \pmod{q^2}$, $g^{\frac{1}{2}q(q-1)} \equiv 1 \pmod{q^2}$.

If a be one solution of $a^p \equiv 1 \pmod{q^2}$, it is obvious that a^2, a^3, ..., a^{p-1} are also solutions, and so are numbers congruous to them. Then, since $3^5 - 1 = 2 \cdot 11^2$; 3, 9, 27, 81 are the four solutions other than 1 of $x^5 \equiv 1 \pmod{11^2}$, $50{,}000 \equiv 27 \pmod{11^2}$; therefore $50{,}000^5 - 1$, is divisible by 11^2. If q be composite $= \alpha^\alpha \beta^\beta \gamma^\gamma \ldots$ a, b, c being prime, $x^p - 1 \equiv \pmod{q^2}$ is satisfied for any value of x prime to q, if $p =$ the L.C.M. of $a^{2a-1}(a-1)$, $b^{2\beta-1}(b-1)$, $c^{2\gamma-1}(c-1)$, ...

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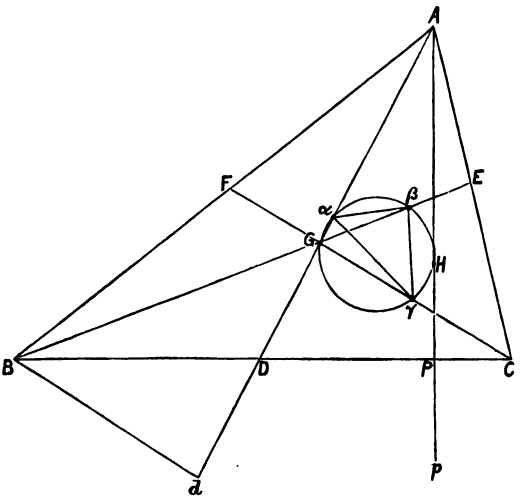
13878. (R. F. Davis, M.A.)—AD, BE, CF are the medians of a triangle ABC co-intersecting in G, the centroid; and H is the orthocentre. Upon DA a point α is taken, so that $DA \cdot D\alpha = DB^2 = DC^2$; similarly for β, γ. Prove that the triangle $\alpha\beta\gamma$ has its sides proportional to the medians of ABC, and that GH is a diameter of its circumcircle.

Solution by W. J. Dobbs, M.A.; R. Tucker, M.A.; *and many others.*

Let AH meet BC in P and the circumcircle in p, so that $HP = Pp$; and draw Bd parallel to FC to meet AD in d.



Then $A\alpha \cdot AD = AD^2 - DA \cdot D\alpha = AP^2 + DP^2 - DC^2 = AP^2 - BP \cdot PC$
$= AP^2 - AP \cdot Pp = AP^2 - AP \cdot PH = AP \cdot AH$;

therefore $H\alpha$ is perpendicular to AD;*

therefore circle on GH as diameter passes through α, and similarly through β and γ. Whence it is easily proved that triangle $\alpha\beta\gamma$ is similar to triangle BdG, *i.e.*, to a triangle whose sides are respectively two-thirds of the medians of ABC.

* Cor.—*AA′ is a chord of a circle bisecting another chord* BC *at* D, *and* AP *is perpendicular to* BC. *The segment* BA′C *is turned about* BC *through half a revolution, and in its new position cuts* AD *at* D′ *and* AP *at* P′. *Then* P′D′ *is perpendicular to* AD.

[See Casey's *Conics*, second edition, page 436, in which the above is described as the *ortho-centroidal* circle.]

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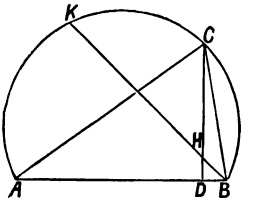
13892. (P. W. Flood.)—Inscribe a triangle in a given segment of a circle, having the sum of the perpendicular and segment of base a maximum.

Solution by H. W. Curjel, M.A.

Let AKCB be the segment. Make angle ABK = ½ a right angle. Bisect arc KCB in C. Then ABC is the required triangle. Draw CD perpendicular to BA, cutting BK and BA in H and D.



Then $AD + DH = AB$,

and CH is a maximum, since the distance from KB is a maximum. Therefore

$CD + AD$ is a maximum.

If BK does not cut the segment, then the sum of the perpendicular and segment of the base is greatest when C coincides with B.

13588. (H. W. Curjel, M.A.)—Prove, by Euclid, Book I., that any triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the triangles, equiangular to it, described on the other two sides.

Solution by the Proposer.

Lemma I.—A rhombus described on the hypotenuse of a right-angled triangle is equal to the sum of the rhombi equiangular to it described on the other two sides.

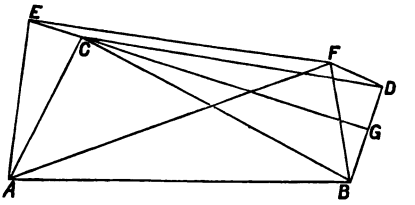


Fig. 1.

Let ABC be a triangle, right-angled at C, and let AE, AC; CD, CB; AF, AB be pairs of adjacent sides of the rhombi, containing equal angles, as in Fig. 1. Let EC meet BD in G. Then $\triangle AEF$ is clearly equal in all respects to $\triangle ACB$. Since the angles AEF, ACB are right angles, the angles DCG, CEF each $= \frac{1}{2} \angle CAE = \frac{1}{2} DCB$; therefore BG = GD, and EF is equal and parallel to CD; therefore ECDF is a parallelogram, and is equal to the triangle FDB; therefore

$$\square ECDF + \triangle ABC = \triangle AEF + \triangle FDB.$$

Taking each of these equals from the whole figure EFDBA,

$$\triangle AEC + \triangle BCD = \triangle ABF.$$

Hence Lemma I.

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Lemma II.—If KH is parallel to LM, and KM = LM, and from Q in LH parallels QR, QP are drawn to LM, LK, cutting HM, HK in R, P, then PR is parallel to KM and equal to QR. Produce RQ to meet KM, KL in S and T.

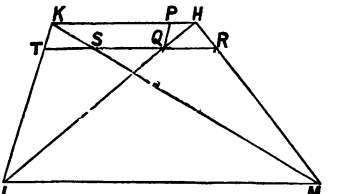


Fig. 2.

Then SR = TQ (CASEY, *Sequel to Euclid*, p. 10, I. 13) = KP, and SR, KP are parallel; therefore KS, PR are equal and parallel; therefore PR is parallel to KM and equal to QR. Hence it is clear that the rhombi drawn on corresponding sides of equiangular triangles and double the triangles respectively are themselves equiangular. Hence the theorem follows by the help of Lemma I. From this and its converse, which may be deduced in the same way as Euclid I. 48 is deduced from I. 47, we can easily prove that, if on the three sides of a right-angled triangle rectilineal figures be described which can be similarly divided into corresponding triangles such that any triangle is equiangular to the corresponding triangles in the other two figures, the figure on the hypotenuse is equal to the sum of the figures on the two other sides.

13751. (Professor SANJÁNA.)—The equation of the ellipse touching the sides of a triangle at the mid-points is $\Sigma\sqrt{(a\alpha)} = 0$. Prove that the centre of this ellipse is at the centroid of the triangle, and that one focus of it is the symmedian point of its own pedal triangle with regard to the original triangle. [This seems the easiest way of obtaining the fundamental property of STEINER's foci.]

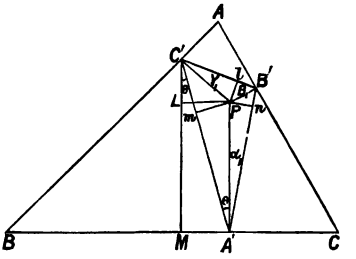
Solution by the Rev. J. CULLEN, B.A.

The first result is obtained by solving for α, β, γ in

$$\frac{1}{a}\frac{df}{da} = \frac{1}{b}\frac{df}{d\beta} = \frac{1}{c}\frac{df}{d\gamma}.$$

The foci satisfy the system

$$\frac{\beta_1\gamma_2 + \beta_2\gamma_1 + 2k^2\cos A}{2U'} = \frac{\gamma_1\alpha_2 + \gamma_2\alpha_1 + 2k^2\cos B}{2V'} = \frac{\alpha_1\beta_2 + \alpha_2\beta_1 + 2k^2\cos C}{2W'}$$



(WHITWORTH's *Modern Geometry*, p. 267), and, since they are isogonal points, we have $\alpha_1\alpha_2 = 1$, $\beta_1\beta_2 = 1$, $\gamma_1\gamma_2 = 1$, $k^2 = 1$,

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Also $U' = 2a^2bc, \quad V' = 2b^2ca, \quad W' = 2c^2ab.$

Now $Pm = \alpha_1 \sin\theta = \alpha_1 MA'/A'C' = \alpha_1 LP/A'C' = \alpha_1\gamma_1 \sin B/A'C';$

therefore $$\frac{A'C'}{Pm} = \frac{\alpha_1^2 + \gamma_1^2 + 2\alpha_1\gamma_1 \cos B}{\alpha_1\gamma_1 \sin B},$$

and, if P is the symmedian point of A'B'C', we have

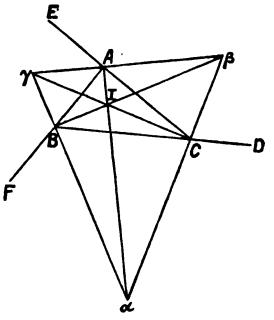
$$\frac{A'B'}{Pn} = \frac{A'C'}{Pm} = \frac{B'C'}{Pl},$$

which is easily seen to be the same as the system for the foci.

13678. (Rev. T. Mitcheson, B.A.)—In the triangle ABC, let AB be produced to F, CA to E, and BC to D; bisect the exterior angles by lines intersecting at α, β, and γ, thus forming the triangles $\alpha\beta\gamma$, BαC, AβC, and AγB. These triangles are equiangular, each angle being the arithmetical mean of the corresponding pair of the angles of ABC.

Solution by W. E. Jeffares, B.A.; L. E. Reay, B.A.; *and many others.*

Bisect the external angles at A, B, C and α, β, γ, where these lines meet, are the centres of the escribed circles opposite A, B, C. Join αA, βB, and γC; these lines bisect the internal angles A, B, C, and meet in I, the centre of inscribed circle. Therefore IBαC, ICβA, IAγC are cyclic quadrilaterals;



therefore $\angle I\alpha B = \angle ICB,$

and $\angle I\alpha C = \angle IBC,$

and $\angle B\alpha C = \frac{1}{2}(B + C),$

$\angle FBC = C + A;$

therefore $\angle \alpha BC = \frac{1}{2}(C + A),$

$\angle ACD = \angle A + \angle B;$

therefore $\angle \beta CD = \frac{1}{2}(A + B).$

Hence the angles of triangle BαC are $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$, and $\frac{1}{2}(A + B)$. Similarly, the angles of the triangle βCA are $\frac{1}{2}(C + A)$, $\frac{1}{2}(A + B)$, $\frac{1}{2}(B + C)$; and those of γAB = $\frac{1}{2}(A + B)$, $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$ respectively; also those of $\alpha\beta\gamma$ are $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$, $\frac{1}{2}(A + B)$. Hence the triangles are similar, and each of their angles is the arithmetical mean of the corresponding pair of the angles of ABC.

13908. (B. N. Cama, M.A.)—With a fixed point as focus and of given magnitude, a series of conics are described. Show that the extremities of all focal chords which are divided by the focus in a fixed ratio lie on either of two fixed circles. [Give a geometrical solution.]

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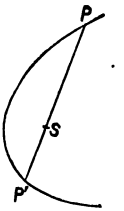
Solution by the PROPOSER.

Let S be the fixed point, and PSP′ one such chord.



Let $SP = n SP'$,

where $n : 1$ is the fixed ratio. Then

$$l = \text{semi-latus rectum} = \frac{2SP \cdot SP'}{SP + SP'} = \frac{2n SP'^2}{(n+1)\,SP'};$$

therefore $$SP' = \frac{n+1}{2n}\, l = \text{constant};$$

therefore locus of P′ is a circle with S as centre. Similarly, the locus of P is a concentric circle.

14040. (C. E. BICKMORE, M.A. See Quest. 13998.)—If u_n be the nth term of the series $1, a, a^2+1, a^3+2a, \ldots$, express u_n as a determinant, and prove the following:—

(1) $u_{2n} = u_n (u_{n+1} + u_{n-1})$, (2) $u_{2n+1} = u_{n+1}^2 + u_n^2$,

(3) $u_{3n} = u_n \{(u_{n+1} + u_{n-1})^2 - (-1)^n\}$, (4) the *proper* factors of u_n (*i.e.*, those which are not factors of u_m, where n is a multiple of m) are of the form $Mn+1$ if they have (a^2+4) as a quadratic residue, and of the form $Mn-1$ if they have (a^2+4) as a quadratic non-residue.

Solution by the PROPOSER.

The series is a recurring series determined by the law

$$u_n = a u_{n-1} + u_{n-2};$$

therefore $$u_n = \begin{vmatrix} a, & 1, & 0, & \ldots \\ -1, & a, & 1, & \ldots \\ 0, & -1, & a, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \text{the ``continuant''}$$

or ``cumulant'' $(a \,.\, a \,.\, a \,.\, a \ldots)$ [to $(n-1)$ terms]. Also

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where} \quad \alpha + \beta = a, \quad \alpha\beta = -1.$$

To prove (1), $$\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \alpha\beta \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) = \alpha^n + \beta^n;$$

therefore $$u_{n+1} + u_{n-1} = \alpha^n + \beta^n \quad \text{and} \quad u_n \ldots = \frac{\alpha^n - \beta^n}{\alpha - \beta};$$

whence $$u_n (u_{n+1} + u_{n-1}) = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = u_{2n}$$

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To prove (2), $\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2-\alpha\beta\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$

$$=\frac{\alpha^{2n+1}(\alpha-\beta)+\beta^{2n+1}\cdot(\beta-\alpha)}{(\alpha-\beta)^2}=\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta};$$

therefore $u_{n+1}^2+u_n^2=u_{2n+1}$.

To prove (3), $\alpha^{2n}+\alpha^n\beta^n+\beta^{2n}=(\alpha^n+\beta^n)^2-(\alpha\beta)^n$;

therefore $\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\alpha-\beta}\left\{(\alpha^n+\beta^n)^2-(\alpha\beta)^n\right\}$;

i.e., $u_{3n}=u_n\left\{(u_{n+1}+u_{n-1})^2-(-1)^n\right\}$.

To prove (4). Let F be any prime factor of u_n; then

$$\alpha^F\equiv\left\{(\tfrac{1}{2}a)^F+[\tfrac{1}{2}\sqrt{(a^2+4)}]^F\right\}\pmod F,$$

by the binomial theorem; and $(\frac{1}{2}a)^F\equiv\frac{1}{2}a\pmod F$; also

$$\left(\frac{a^2+4}{4}\right)^{\frac{1}{2}(F-1)}\equiv\frac{(a^2+4)^{\frac{1}{2}(F-1)}}{2^{F-1}}\equiv\pm1\pmod F.$$

(i.) Let a^2+4 be a quadratic residue of F; therefore

$(a^2+4)^{\frac{1}{2}(F-1)}\equiv+1\pmod F$; therefore $\alpha^F\equiv\alpha\pmod F$,

similarly, $\beta^F\equiv\beta\pmod F$; whence $\alpha^{F-1}-\beta^{F-1}\equiv0\pmod F$.

Let $F-1=Mn$, where M is 1 or any positive integer, therefore

$$u_n\equiv0\ [\mathrm{mod}\ (Mn+1)].$$

(ii.) Let a^2+4 be a quadratic non-residue of F,

$$(a^2+4)^{\frac{1}{2}(F-1)}\equiv-1\pmod F,$$

and $\alpha^F\equiv\beta\pmod F$, $\beta^F\equiv\alpha\pmod F$;

therefore $\alpha^{F+1}\equiv\beta^{F+1}\equiv\alpha\beta\equiv-1\pmod F$;

therefore $\alpha^{F+1}-\beta^{F+1}\equiv0\pmod F$.

Let $F+1=Mn$, where M is 1 or any positive integer; therefore

$$u_n\equiv0\ [\mathrm{mod}\ (Mn-1)].$$

13802. (R. Chartres.)—Sum to infinity

$$1+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1.3}{2.4}+\frac{1}{4}\cdot\frac{1.3.5}{2.4.6}+\&c.$$

Solution by Rev. W. Allen Whitworth, M.A.; Professor Sanjána, M.A.; *and others.*

We have (*DCC Exercises*, p. xviii.)

$$1+\frac{Y_1x}{2}+\frac{Y_2x^2}{3}+\frac{Y_3x^3}{4}+\ldots=\frac{1-\sqrt{(1-4x)}}{2x},$$

where $Y_n\equiv\frac{2n!}{n!\,n!}$ or $\frac{Y_n}{4^n}=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$.

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Hence, writing $\frac{1}{4}$ for x, we obtain

$$1+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1.3}{2.4}+\frac{1}{4}\cdot\frac{1.3.5}{2.4.6}+\ldots=2,$$

the sum required.

Observe that the given series is a particular case (when $p=1$ and $x=\frac{1}{4}$) of the more general theorem

$$\frac{1}{p}+\frac{x\mathrm{R}_1^{p+1}}{p+1}+\frac{x^2\mathrm{R}_2^{p+2}}{p+2}+\frac{x^3\mathrm{R}_3^{p+3}}{p+3}+\ldots=\frac{1}{p}\left\{\frac{1-\sqrt{(1-4x)}}{2x}\right\}^p,$$

where R_r^n denotes, as usual, the number of combinations of n things r together, when repetitions are allowed.

[Mr. Curjel observes that the given series is convergent, and $=\int_0^1(1-x)^{-\frac{1}{2}}\,dx=2.$]

13816. (H. W. Curjel, M.A.)—Find the form of the function f such that

$$f(a+x)+f(b+x)-f(a+x)\,f(b+x)=f(c+x)$$

where c depends only upon a and b. Also determine c as a function of a and b.

Solutions by (1) Professor S. Sircom, M.A; (2) Rev. J. Cullen, B.A.

(1) The solution is $f(x)=1-m^{n^x}$ where m, n are constants and $n^c=n^a+n^b$, as may be easily verified or proved as follows:—

Putting x for $x+a$, d for $b-a$, then $c-a$ is a function of d, and we may write $\{1-f(x)\}\{1-f(x+d)\}=1-f\{x+\phi(d)\}$.

Putting $x=0$, d, and $\phi(d)$, &c., we have

$$\{1-f(0)\}\{1-f(d)\}=1-f\{\phi(d)\},$$
$$\{1-f(0)\}\{1-f(d)\}^2\{1-f(2d)\}=1-f\{2\phi(d)\},$$
$$\{1-f(0)\}\{1-f(d)\}^3\{1-f(2d)\}^3\{1-f(3d)\}=1-f\{3\phi(d)\}.$$

Putting $d=0$,

$$\{1-f(0)\}^2=1-f\{\phi(0)\},\quad\{1-f(0)\}^{2^2}=1-f\{2\phi(0)\},$$

and, generally, $\{1-f(0)\}^{2^p}=1-f\{p\phi(0)\}$.

$f(0)$ and $\phi(0)$ being constants, we have

$$1-f(pl)=m^{2^p},\quad 1-f(x)=m^{n^x}.$$

(2) Writing the equation in the form

$$\{1-f(a+x)\}\{1-f(b+x)\}=\{1-f(c+x)\},$$

we have $u_{a+x}+u_{b+x}=u_{c+x}$ where $u_x=\log\{1-f(x)\}$;

therefore $f(x)=1-e^{\Sigma C\lambda_r^x}$,

λ_r being a root of $\lambda^{a-c}+\lambda^{b-c}-1=0$.

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Also $$u'_{c+x}\frac{dc}{da} = u'_{a+x},\ u'_{c+x}\frac{dc}{db} = u'_{b+x},$$

and, differentiating with respect to x, we get

$$u'_{a+x} + u'_{b+x} = u'_{c+x};\quad \text{therefore}\quad \frac{dc}{da} + \frac{dc}{db} = 1.$$

Hence $$2c = a + b + \phi(a-b),$$

ϕ being any arbitrary function. In the particular case when $2c = a+b$, we may put $$\Sigma C\lambda_r^x = \Sigma C \sin(2x\theta_r + \gamma),$$

where $$\theta_r = (2r + \tfrac{1}{3})\pi/(a-b).$$

13840. (Professor A. Droz-Farny.)—On considère une circonférence fixe C et une circonférence variable C′ qui reste constamment tangente à deux droites donnés. Quelle est l'enveloppe de l'axe radical des circonférences C et C′?

Solution by R. Tucker, M.A.; L. E. Reay, B.A.; *and others.*

The equation to the tangential circle may be written

$$S = 2\lambda(x+y) - \lambda^2,$$

and to the given circle $S = ax + by + c$;

therefore the envelope is the parabola

$$(x+y)^2 = ax + by + c.$$

13737. (The late K. A. H. Fischer.)—Solve the equations:—

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1;\quad x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 = 0;$$
$$x_1y_1^2 + \ldots = \tfrac{1}{3};\quad x_1y_1^3 + \ldots = 0;\quad x_1y_1^4 + \ldots = \tfrac{1}{5};\quad x_1y_1^5 + \ldots = 0;$$
$$x_1y_1^6 + \ldots = \tfrac{1}{7};\quad x_1y_1^7 + \ldots = 0;\quad x_1y_1^8 + \ldots = \tfrac{1}{9};\quad x_1y_1^9 + \ldots = 0.$$

Solution by H. W. Curjel, M.A.

Let the ys be the roots of

$$t^5 + at^4 + bt^3 + ct^2 + dt + e = 0.$$

Multiplying five consecutive equations in order by e, d, c, b, a, and 1, and adding, we get $$e + \tfrac{1}{3}c + \tfrac{1}{5}a = 0;$$

$\tfrac{1}{3}d + \tfrac{1}{5}b + \tfrac{1}{7} = 0$; $\tfrac{1}{3}e + \tfrac{1}{5}c + \tfrac{1}{7}a = 0$; $\tfrac{1}{5}d + \tfrac{1}{7}b + \tfrac{1}{9} = 0$; $\tfrac{1}{5}e + \tfrac{1}{7}c + \tfrac{1}{9}a = 0$;

therefore $$a = c = e = 0,\quad b = -\tfrac{10}{9};\quad d = \tfrac{5}{21};$$

therefore the ys are the roots of $t(t^4 - \frac{10}{9}t^2 + \frac{5}{21}) = 0$. Also it is easily shown that no two ys can be equal; hence the xs may easily be found.

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13783. (Professor UMES CHANDRA GHOSH.)—Prove that

$$\int_0^\infty \frac{\sin(n+1)x \,.\, \sin nx}{x \,.\, \sin x}\, dx = \tfrac{1}{2}(n\pi).$$

Solution by LIONEL E. REAY, B.A.; R. CHARTRES; *and many others.*

$$\int_0^\infty \frac{\sin(n+1)x \,.\, \sin nx}{x \,.\, \sin x}\, dx = \int_0^\infty \frac{\sin 2x + \sin 4x + \ldots + \sin 2nx}{x}\, dx$$

$$= n\int_0^\infty \frac{\sin 2nx}{x}\, dx = \tfrac{1}{2}n\pi.$$

5108. (S. A. RENSHAW.)—Show that, if any side of a self-conjugate triangle inscribed in a conic be produced to meet the directrix, the line joining the point of intersection therewith with the focus is at right angles to the line joining the focus with the opposite vertex of the triangle.

Solution by Professor SANJÁNA, M.A.

Let A, B, C be the vertices of the self-conjugate, and F the focus; let BC meet the directrix in D. Since BC is the polar of A, and the directrix is the polar of F, therefore D is the pole of AF. Thus AF is the focal chord of contact of tangents from D, and hence the angle DFA is a right angle.

13972. (Rev. W. ALLEN WHITWORTH, M.A.)—If a die of p faces be thrown n times, the expectation of the lowest throw is $1+k$ and the expectation of the highest is $p-k$, where

$$k = \left\{1^n + 2^n + 3^n + \ldots + (p-1)^n\right\} \div p^n.$$

Solution by H. W. CURJEL, M.A.

$$\text{Expectation of lowest throw} = \Sigma_1^p\, x\, \frac{(p-x+1)^n - (p-x)^n}{p^n} = k+1.$$

$$\text{Expectation of highest throw} = \Sigma_1^p\, \frac{x\left\{x^n - (x-1)^n\right\}}{p^n} = p-k.$$

The sum of these expectations $= p+1$, which might have been inferred *a priori*.

13796. (C. E. BICKMORE, M.A.)—Solve the difference-equations

(1) $u_n^4 + 1 = (u_{n-1}u_n + 1)(u_n u_{n+1} + 1)$,

(2) $u_n^4 - u_n^2 + 1 = (u_{n-1}u_n + 1)(u_n u_{n+1} + 1)$.

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Solution by the PROPOSER.

The assumption $u_{n+1} + u_{n-1} = ku_n$

reduces (1) to $u_n^2 = u_{n-1}u_{n+1} + k$, and (2) to $u_n^2 - 1 = u_{n-1}u_{n+1} + k$;

therefore $u_n = A\alpha^n + B\beta^n$ where $\alpha + \beta = k$, $\alpha\beta = 1$,

which satisfies (1) if $AB(\alpha - \beta)^2 = -k$,

and (2) if $AB(\alpha - \beta)^2 = -(k+1)$.

Since $\beta = 1/\alpha$, $k = \alpha + 1/\alpha$,

while in (1). $B = -(\alpha + 1/\alpha)/A \div (\alpha - 1/\alpha)^2$,

and in (2) $B = (\alpha + 1 + 1/\alpha)/A \div (\alpha - 1/\alpha)^2$,

the solutions involve only the two arbitrary constants α and A.

A simple solution, giving u_n an integer, is for (1)

$u_1 = x$, $u_2 = x^3$, $u_3 = x^5 - x$, $k = x^2$, $u_n = x(\alpha^n - \beta^n) \div (\alpha - \beta)$,

where $\alpha + \beta = x^2$ and $\alpha\beta = 1$; and for (2)

$u_1 = x$, $u_2 = x^3 - x$, $u_3 = x^5 - 2x^3$, $k = x^2 - 1$, $u_n = x(\alpha^n - \beta^n) \div (\alpha - \beta)$,

where $\alpha + \beta = x^2 - 1$, and $\alpha\beta = 1$.

By a similar method, one may solve

$$(u_n^4 + 1) + a(u_n^3 + u_n) + bu_n^2 = (1 - u_{n-1}u_n)(1 - u_n u_{n+1}).$$

13824. (R. KNOWLES, B.A.)—P, P′ are any two points on a parabola; the tangent at P meets the diameter through P′ in Q. Prove that the tangent at P′ bisects PQ. [See Quest. 13372, and BESANT, *Conics*, Prop. XVII., § 45.]

Solution by Rev. T. MITCHESON, B.A.; Professor A. DROZ-FARNY; *and others.*

It is well known that the tangents at the extremities of any chord PP_1 meet in the diameter which bisects this chord, but, by similar triangles, this diameter also bisects PQ, &c.

13858. (ROBERT W. D. CHRISTIE.)—Prove the following:—Where P is any prime whatever, $4P + 5$ equals one, two, or three integral squares. Distinguish the three cases.

Solutions (1) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(2) *by the* PROPOSER.

(1) (i.) The condition that $(4P + 5) = \square = a^2$ is $P = \frac{1}{4}(a^2 - 5)$.

(ii.) The condition that $(4P + 5) = a^2 + b^2$ is that $4P + 5 =$ prime (p) of form $p = 4\varpi + 1$, or = product of several such primes.

(iii.) If neither of above conditions be fulfilled, then

$$(4P + 5) = a^2 + b^2 + c^2;$$

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for it is shown by LEGENDRE, *Théorie de Nombres*, Art. 317 (v. 1), that every odd number, except those of form $(8n+7)$, is the sum of three squares.

(2) (i.) Let $P = 2T_n - 1$; then $4P+5 = (2n+1)^2$, where T_n means a triangular with root n.

(ii.) Let $P = T_n + T_m - 1$; then $4P+5 = (n+m+1)^2 + (n-m)^2$.

(iii.) In all *other* cases, $4P+5 = 4(6N \pm 1)+5 = 8M+1$,

and therefore equals three integral squares by long established results.

[Mr. WHITWORTH remarks as follows:—Is not the proposition true whether P be a prime or not? Is LEGENDRE's exception admissible here? For every number is either an ace, or a deuce, or a terce, or a quart (*i.e.*, the sum of one, two, three, or four integral squares). But $4N+1$ is never a quart. And the cases are distinguished as follows:—Let $4N+1$ be resolved into its prime factors, and let all factors occurring an even number of times be cast out. If no factors remain, $4N+1$ is an ace. If the factors remaining are all of the form $4n+1$, then $4N+1$ is a deuce. If there remain any factors of the form $4n-1$, then $4N+1$ is a terce.

To this Lt.-Col. ALLAN CUNNINGHAM adds the following:—"The PROPOSER's restriction that P should be *prime* is unnecessary, and is, in fact, not used in the solution. The clause, 'except those of form $(8n+7)$,' is a necessary part of LEGENDRE's theorem. These excepted numbers cannot of course occur in the present question, which is limited to those of form $(4P+5)$."]

13942. (R. F. DAVIS, M.A.)—Prove that from the relation

$$(x+y+z+p)(x^{-1}+y^{-1}+z^{-1}+q^{-1}) = \lambda$$

there may be deduced three equations:

$$\left.\begin{aligned} a(yz)^{-1} + byz + c + f(y+z) + g(y^{-1}+z^{-1}) + h(y/z+z/y) &= 0 \\ a(zx)^{-1} + \ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ &= 0 \\ a(xy)^{-1} + \ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ &= 0 \end{aligned}\right\} \text{(A)},$$

where $h^2 - ab - ch + fg = 0$. Conversely, ascertain whether from three equations of the form (A) the above relation may be inferred.

Solution by the PROPOSER.

From the given relation, we can obtain, by multiplying up and arranging terms,

$$(y/z+z/y)+(x+y+z+p)/x+p(1/y+1/z)+(y+z)/q + (yz+zx+xy+xyz/q)/yz+p/q-\lambda+1 = 0.$$

Writing a/h for $xyz(1/x+1/y+1/z+1/q)$, b/h for $(x+y+z+p)/xyz$, c/h for $p/q-\lambda+1$, f/h for $1/q$, g/h for p, we have the first equation. Since the above values are *symmetrical*, the second and third equations might equally well have been deduced. Observe $ab/h^2 = \lambda$ and $c/h = (fg-ab)/h^2+1$.

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Conversely the system (A) with condition subsisting as regards constants implies two symmetrical relations between all three variables

$$\left.\begin{aligned} x+y+z+g/h &= a/h\,xyz \\ 1/x+1/y+1/z+f/h &= (b/h)\,xyz \end{aligned}\right\}.$$

[See the PROPOSER's paper on "Porismatic Equations," read before the Mathematical Association on Jan. 21st, 1899.]

13880. (Rev. W. ALLEN WHITWORTH, M.A.)—Find three numbers (each less than 10,000) whose squares are in arithmetical progression, so that the ratio of the greatest to the least may be (i.) as great as possible, and (ii.) as small as possible.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let x, y, z be the three numbers; and $x>y>z$, and $x<10{,}000$. Then

$$x^2-y^2 = y^2-z^2; \qquad \text{therefore} \quad x^2 = 2y^2-z^2 = (2y-z)^2-2\,(y-z)^2.$$

The *general* solution of this in terms of *two* arbitrary numbers m, n is known to be $x = m^2 \sim 2n^2$, $2y-z = m^2+2n^2$, $y-z = 2mn$,

whence $x = m^2 \sim 2n^2$, $y = m^2-2mn+2n^2$, $z = m^2-4mn+2n^2$.

Hence the ratio of the greatest (x) to the (leas) z is

$$\frac{x}{z} = \frac{m^2 \sim 2n^2}{m^2-4mn+2n^2},$$

to be a minimum or a maximum.

Since $x>z$, the *least* value is when $x+z = 1$ nearly. This least $x+z$ is given by one of the following cases:—(1) $n = 1$, $m =$ greatest value, such that $x<10{,}000$; this gives $m = 100$. Here

$$x = 9998, \quad y = 9802, \quad z = 9602; \qquad x+z = 1{\cdot}041.$$

(2) $m = 1$, $n =$ greatest value, such that $x<10{,}000$; this gives $n = 70$, Here $x = 9799$, $y = 9661$, $z = 9521$; $x+z = 1{\cdot}029$.

[The values $m = n \pm 1$, $m = 2n \pm 1$ also lead to minima of the ratio $x+z$, but they involve some of x, y, z *negative;* these are excluded, because *all* such values are $<10{,}000$.]

The maximum value of the ratio $x : z$ may be found by observing that $z = \sqrt{(2y^2-x^2)}$; for this gives x^2, y^2, z^2 in arithmetical progression. The maximum value of $x+z$ is given by $z^2 = 2y^2-x^2 = 1$. The solution of this Diophantine gives the following values of x, y,

$$x = 1, 7, 41, 239, 1393, 8119, 47321, \&c.,$$

$$y = 1, 5, 29, 169, 985, 5741, 33461, \&c.,$$

so that the values of x, y, z ($<10{,}000$) giving the maximum value of $x : z$ are 8119, 5741, 1.

The above also shows that 1, 1, 1 are the values giving the *absolutely least* value of $x+z = 1$ (if an arithmetical progression with *zero* common difference be admitted).

[The PROPOSER sends the following answer:—

(i.) 1, 5741, 8119; (ii.) 9521, 9661, 9799.]

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13829. (R. CHARTRES.)—If from the vertex A of a triangle there be drawn to the base the altitude AD, the bisector of the angle AE, and the median AF, then, if DE = 3EF, the sides of the triangle will be in arithmetical progression.

Solution by C. JOSS, M.A.; Prof. T. SAVAGE; *and many others.*

Draw BGH perpendicular to AE. Then GF is parallel to, and half of HC, the difference of the sides; therefore

$$EGF = \tfrac{1}{2}A,$$

and, because ABDG is concyclic,

$$GDE = \tfrac{1}{2}A;$$

therefore FG is tangent to circumcircle of GDE; therefore

$$DF \cdot FE = GF^2;$$

therefore $2EF = GF$,

and $$\frac{BE}{EC} = \frac{BA}{AC}; \quad \therefore \quad \frac{BE+EC}{BE-EC} = \frac{BA+AC}{BA-AC};$$

$$\therefore \quad \frac{BC}{2EF} = \frac{BA+AC}{2GF}; \quad \therefore \quad 2BC = BA+AC.$$

14042. (Prof. G. B. MATHEWS, M.A.)—Prove that

$$a(b-c)^{6n+1} + b(c-a)^{6n+1} + c(a-b)^{6n+1}$$

is divisible by $(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$

for all positive integral values of n. Thus, when $n = 1$, the quotient is

$$(a+b+c)(a^2+b^2+c^2+bc+ca+ab) - 18abc.$$

Is there any convenient way of finding the quotients for higher values of n?

Solution by J. O. WATTS; L. E. REAY, B.A.; *and others.*

$$a(b-c)^{6n+1} + b(c-a)^{6n+1} + c(a-b)^{6n+1}$$

is clearly divisible by $(a-b)$, $(b-c)$, $(c-a)$, for it vanishes when $a = b$, $b = c$, $c = a$.

Put $$a = -\omega b - \omega^2 c, \quad -\omega(b+\omega c)(b-c)^{6n+1} + b(c+\omega b+\omega^2 c)^{6n+1} + c(-\omega^2 b - \omega^2 c - b)^{6n+1}$$

$$= -\omega(b+\omega c)(b-c)^{6n+1} + \omega b(b-c)^{6n+1} + \omega^2 c(b-c)^{6n+1} = 0.$$

Therefore $a+\omega b+\omega^2 c$ is a factor. Similarly $a+\omega^2 b+\omega c$ is a factor.

Therefore $a^2+b^2+c^2-ab-bc-ca$ is a factor.

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13950. (Ernest P. Barrett, B.A.)—If O be the point from which two common tangents are drawn to two orthogonal circles, whose points, A and C, of intersection are fixed, prove that, if the intersections of the polars of O with respect to the circles with the join of O to the centres are B and D, ABCD is a square.

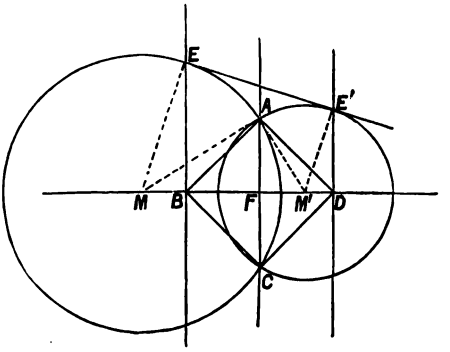
Solution by Professor A. Droz-Farny; H. W. Curjel, M.A.; *the* Proposer; *and* F. H. Peachell, B.A.

Soient M et M′ les centres des deux circonférences de rayons R et R′, E et E′ les points de contact d'une des tangentes communes, d la distance, M, M′ des centres et F le point milieu de AC. On a :

$$(MA)^2 = MF \cdot MM' \quad \text{et} \quad (M'A)^2 = M'F \cdot MM';$$

donc $$R^2R_1^2 = MF \cdot M'F \cdot d^2 = (AF)^2 d^2,$$

et par conséquent $$2RR' = d \cdot AC.$$



On trouve aisément que

$$d \cdot BD = (EE')^2 = d^2 - (R-R')^2 = R^2 + R_1^2 - (R-R_1)^2 = d \cdot AC,$$

d'où $$BD = AC.$$

Mais l'axe radical des deux circonférences divise EE′, donc aussi BD, en parties égales, et par conséquent BF = FD = FA = FC. ABCD est un carré.

13879. (J. Brill, M.A.)—Verify that the six quantities $x, y, z, \xi, \eta, \zeta$ can be eliminated from the six equations

$$a + dx + e\xi = 0, \quad b + dy + e\eta = 0, \quad c + dz + e\zeta = 0,$$
$$f_1 + g_2 z - g_3 y - h_2\zeta + h_3\eta + k(y\zeta - z\eta) = 0,$$
$$f_2 + g_3 x - g_1 z - h_3\xi + h_1\zeta + k(z\xi - x\zeta) = 0,$$
$$f_3 + g_1 y - g_2 x - h_1\eta + h_2\xi + k(x\eta - y\xi) = 0,$$

and obtain the result of elimination.

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Solution by H. W. CURJEL, M.A.; *and the* PROPOSER.

Substituting in the last three equations for ξ, η, ζ from the first three, we get

$$F_1 + yG_3 - zG_2 = 0, \quad F_2 - G_3x + G_1z = 0, \quad F_3 + G_2x - G_1y = 0 \quad (1, 2, 3),$$

where
$$F_1 = -f_1 - \frac{h_2c}{e} + \frac{h_3b}{e}, \quad G_1 = g_1 + \frac{h_1d + ka}{e},$$
$$F_2 = -f_2 + \frac{h_1c - h_3a}{e}, \quad G_2 = g_2 + \frac{h_2d + kb}{e},$$
$$F_3 = -f_3 + \frac{h_2a - h_1b}{e}, \quad G_3 = g_3 + \frac{h_3d + kc}{e}.$$

Multiplying (1) by G_1, (2) by G_2, (3) by G_3 and adding, we get

$$F_1G_1 + F_2G_2 + F_3G_3 = 0;$$

i.e.,
$$e(f_1g_1 + f_2g_2 + f_3g_3) + d(f_1h_1 + f_2h_2 + f_3h_3) + ka(f_1 + f_2 + f_3)$$
$$= a(g_3h_2 - g_2h_3) + b(g_1h_3 - g_3h_1) + c(g_2h_1 - g_1h_2)$$
$$= \begin{vmatrix} a, & h_1, & g_1 \\ b, & h_2, & g_2 \\ c, & h_3, & g_3 \end{vmatrix}$$

13939. (C. E. HILLYER, M.A.)—AOB is the diameter of a circle of radius R; circles are described on OA and OB as diameters; a circle O_1 is then described, touching the last two externally and the original circle internally; a second circle O_2 touches O_1 and the circle on OA externally, and the original circle internally; and so on. Show that the angles AOO_2, AOO_3, AOO_4, ... are successive acute angles in descending order which have all their trigonometrical ratios rational, and the radius of the nth circle O_n is $\frac{R}{n^2+2}$. Also, if O be not the centre, but $OA = a$ and $OB = b$, then the diameter of the nth circle, similarly constructed, is $\frac{ab(a+b)}{n^2b^2 + a(a+b)}$. [See, regarding the second part, Quest. 7905, by R. LACHLAN, B.A., Vol. LXI., p. 94.]

Solution by the PROPOSER *and* Professor SANJÁNA, M.A.

If r_n, θ_n are the polar coordinates of the centre of O_n, and a_n the radius of the nth circle, we have
$$r_n + a_n = R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$
and
$$\sqrt{(r_n^2 + \tfrac{1}{4}R^2 - r_n R\cos\theta_n)} - a_n = \tfrac{1}{2}R;$$
therefore
$$r_n + \sqrt{(r_n^2 + \tfrac{1}{4}R^2 - r_n R\cos\theta_n)} = \tfrac{3}{2}R,$$
whence
$$\cos\theta_n = 3 - 2(R/r_n) \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

Since $\theta_1 = 90°$, we have $R/r_1 = \frac{3}{2}$. Now, if we form the equation for r_2 from the condition that the circle O_2 touches the three circles O_1, that

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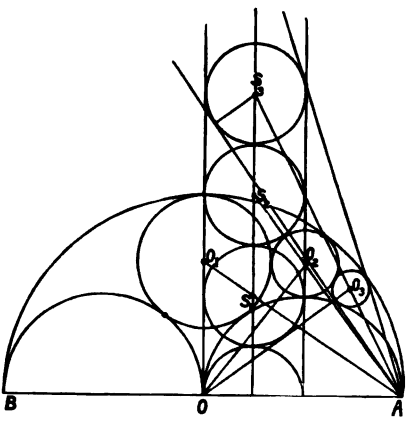
on OA_1, and that on AB, we shall obtain a quadratic of which $\frac{1}{2}R$ is one root, since the circle on OB also satisfies the condition [as a matter of fact the equation is $12r^2-16Rr+5R^2=0$ or $(2r-R)(6r-5R)=0$]; therefore R/r_2 is rational. Similarly, since R/r_1 is rational, it follows that R/r_3 is so also, and so on, and therefore, from (2), $\cos\theta_n$ is always rational, and, from (1), R/a_n is always rational. Now, from the triangle $O_{n-1}O\,O_n$, we have

$$\cos(\theta_{n-1}-\theta_n) = \frac{r_n^2+r_{n-1}^2-(a_n+a_{n-1})}{2r_nr_{n-1}} = \frac{-2R^2+2Rr_n+2Rr_{n-1}-r_nr_{n-1}}{r_nr_{n-1}}$$

which is rational; and, therefore,

$$\sin\theta_n\sin\theta_{n-1}$$

is rational, but $\sin\theta_1=1$; therefore $\sin\theta_2$ is rational; therefore $\sin\theta_3$ is rational, and so on.



Thus θ_1, θ_2, θ_3, &c., are successive angles having both sine and cosine rational.

Now $(n^2-1)^2+(2n)^2=(n^2+1)^2$ and we may take

$$\cos\theta_n=\frac{n^2-1}{n^2+1},\qquad \sin\theta_n=\frac{2n}{n^2+1},\qquad \frac{R}{r_n}=\tfrac{1}{2}(3-\cos\theta_n)=\frac{n^2+2}{n^2+1},$$

$$a_n=R-r_n=\frac{R}{n^2+2}.$$

The last result is obtained shortly by inversion, thus:—

Invert with respect to the circle whose centre is A and radius AO; then the circle on AO inverts into the straight line OO_1, and that on AB into a parallel line through the centre of the other. Thus the circles O_1, O_2, O_3, ... invert into equal circles, each of radius $\frac{1}{4}R$, touching each other and these two parallels. Let S_n be the centre of the circle which is the inverse of O_n,

$$\frac{a_n}{\frac{1}{4}R}=\frac{R^2}{AS_n^2-\frac{1}{16}R^2}.$$

But

$$AS_n^2=\left(\frac{nR}{2}\right)^2+\left(\frac{3R}{4}\right)^2;$$

therefore

$$AS_n^2-\frac{R^2}{16}=\frac{(n^2+2)R^2}{4};\qquad \text{therefore}\quad a_n=\frac{R}{n^2+2}.$$

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This can easily be extended to the case when O is not the centre,

when $$a_n = \tfrac{1}{2}\frac{ab\,(a+b)}{n^2b^2 + a\,(a+b)},$$

where $$a + b = 2\mathrm{R}.$$

Note.—Since proposing this question I have discovered that the last part is the same as Quest. 7905 by Dr. LACHLAN. Also the angles AOO_2, AOO_3, &c., and their complements do not include *all* the angles whose ratios are rational, *e.g.*, $\sin^{-1}\frac{8}{17}$ can only be obtained from the formula $\sin\theta_n = 2n/(n^2+1)$ by putting $n = \frac{4}{1}$ or $\frac{1}{4}$. The general formula is $2nm/(n^2+m^2)$. The question arises, Can the other angles obtained by giving to m values other than 1 be constructed by means of tangential circles?

13979. (R. CHARTRES.)—Show that

$$\int_0^\infty \frac{x^{k-1}\,.\,dx}{1+x} = \frac{\pi}{\sin k\pi}$$

without DEMOIVRE'S theorem or imaginaries ($k < 1$); evaluate

$$\int_0^\infty \frac{\log x\,.\,dx}{1+x^3};$$

and sum the series $$\frac{1}{1^2\,.\,2^2} - \frac{3}{4^2\,.\,5^2} + \frac{5}{7^2\,.\,8^2} - \&c.,\ \textit{ad infinitum}.$$

Solution by Professor NANSON, G. H. HARDY, *and the* PROPOSER.

(1) $$\int_0^\infty \frac{x^{k-}}{1+x}\,dx = \int_0^1 \frac{x^{k-1}+x^{-k}}{1+x}\,dx = \pi\,\mathrm{cosec}\,k\pi,$$

on expanding $(1-x)^{-1}$ in a G.P., integrating each term separately, and summing by the known formula

$$\pi\,\mathrm{cosec}\,k\pi = \frac{1}{k} + \sum_{r=1}^{r=\infty}\left(\frac{1}{k+r} + \frac{1}{k-r}\right)(-1)^r,$$

which can be proved without using DEMOIVRE'S theorem, or imaginary quantities.

(2) $$\begin{aligned}\int_0^\infty \frac{\log x}{1+x^3}\,dx &= \int_0^1 \frac{1-x}{1+x^3}\log x\,dx \\ &= \int_0^1 \left\{1-x-x^3+x^4+x^6-x^7-\ldots\right\}\log x\,dx \\ &= -\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{8^2}+\ldots \\ &= -\frac{2\pi^2}{27},\ \text{as in next part.}\end{aligned}$$

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(3) Let $$S = \frac{1}{1^2.2^2} - \frac{3}{4^2.5^2} + \frac{5}{7^2.8^2} - \dots;$$

therefore $$3S = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \dots.$$

Now $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{8^2} - \dots;$$

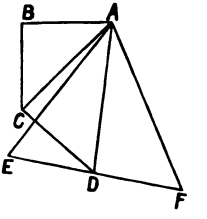
therefore $$3S - \frac{\pi^2}{12} = -\frac{1}{9}\frac{\pi^2}{12}; \quad \text{therefore} \quad S = \frac{2\pi^2}{81}.$$

[The PROPOSER remarks that (2) follows at once from (1) by differentiating with respect to k and putting $k = \frac{1}{3}$.]

14020. (SALUTATION.)—From a given point as vertex construct an equilateral triangle without describing any arc, but having a flat ruler with two of its edges at right angles and half the length of the required side set off on one of them, measured from their point of junction. [Not only is the ordinary joining of points postulated, but the apposition of either of the two specified points on the ruler to a point in the figure, as it grows, and of one of the edges to any line already drawn; also the drawing of a new line along the other edge, whilst the position of the ruler is maintained.]

Solution by G. W. PRESTON.

Let A be the given vertex. Place the corner of the ruler at A, and draw a line from A equal to the half-side. Place the corner of the ruler at B, the other extremity of this line, and in such a way that the unmarked edge coincides with the line AB in direction. Draw BC along the other edge equal to the half-side. Join AC. Place the ruler again at C, and so that the unmarked edge coincides with CA, and draw CD along the other edge equal to the half-side.



Then $$AD^2 = 3a^2,$$

where $2a$ is the given side. Repeat the same process on each side of AD, then drawing DE, DF each equal to the half-side. Join AE, AF. Then AEF is the equilateral triangle required.

For $$AF^2 = AD^2 + DF^2 = 3a^2 + a^2 = 4a^2.$$

Therefore $$AF = 2a.$$

Similarly, $$AE = 2a,$$

and EF also equals $2a$.

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13973. (R. KNOWLES, B.A.)—$y^2 = 4ax$ is the equation to a parabola vertex A: on the tangent at A, AT is taken $= 4a$; from T, TP is drawn to touch the curve in P; the chord AQ is drawn at right angles to AP; the lines joining the pole of AP to Q and that of AQ to P intersect in O. Prove (1) that the equation to the polar of O is $21x + 6y + 16a = 0$; (2) PQ meets this polar on AT.

Solutions by F. H. PEACHELL, B.A.; G. W. PRESTON, B.A.; *and others.*

(1) Coordinates of P are $16a$, $8a$. Equation of AP is

$$y/8a = x/16a, \quad \text{or} \quad 2y - x = 0.$$

Therefore equation of AQ is

$$2x + y = 0,$$

from which we get coordinates of Q, a, $2a$.

Tangent at Q is

$$-2ay = 2a(x + a);$$

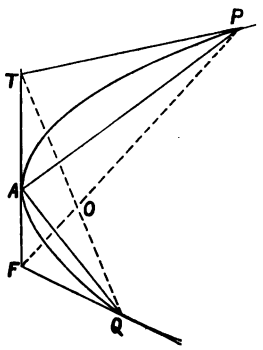
where this cuts $x = 0$, $y = -a$.

Therefore F is point $0, -a$.

Equation of TQ is

$$6x + y - 4a = 0;$$

FP is $9x - 16y - 16a = 0$.



Therefore coordinates of O are $\frac{16}{21}a$, $-\frac{12}{21}a$. Therefore polar of O is

$$(-y \,.\, 12a)/21 = 2a(x + \tfrac{16}{21}a), \quad \text{or} \quad 21x + 6y + 16a = 0.$$

(2) Equation of PQ is $3y - 2x + 8a = 0$;

polar of O is $21x + 6y + 16a = 0$.

These evidently meet on $x = 0$, that is, on AT.

13927. (Professor SANJÁNA.)—Prove the identity

$$(x^{2n+1} - y^{2n+1})/(x - y) = (x^n + x^{n-1}y + \ldots + y^n)^2 - xy(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})^2;$$

and show how it may be employed to obtain factors of large numbers of the form $x^{4n+2} - 1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and* H. W. CURJEL, M.A.

Let $P = (x^n + x^{n-1}y + \ldots y^n)$, $Q = (x^{n-1} + x^{n-2}y + \ldots + y^{n-1})$.

Then $$P = \frac{x^{n+1} - y^{n+1}}{x - y}, \quad Q = \frac{x^n - y^n}{x - y};$$

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$$\therefore\quad P^2 - xy\,.\,Q^2 = \frac{(x^{2n+2} - 2x^{n+1}y^{n+1} + y^{2n+2}) - xy\,(x^{2n} - 2x^n y^n + y^{2n})}{(x-y)^2},$$

$$= \frac{x^{2n+2} - x^{2n+1}y - xy^{2n+1} + y^{2n+2}}{(x-y)^2}$$

$$= \frac{x^{2n+1}(x-y) - y^{2n+1}(x-y)}{(x-y)^2} = \frac{x^{2n+1} - y^{2n+1}}{x-y}.$$

[This is a *known* theorem. It is given in Mr. BICKMORE's paper "On the Numerical Factors of $(a^n - 1)$," in *Messenger of Mathematics*, Vol. XXVIII., Art. 8.]

Next, let $x = \xi^2$, $y = 1$. Then

$$\frac{\xi^{4n+2} - 1}{\xi^2 - 1} = P^2 - \xi^2\,.\,Q^2 = (P - \xi Q)(P + \xi Q),$$

thus giving two factors of $(\xi^{4n+2} - 1)$, apparently *new*. But, on substituting for P, Q in terms of ξ, it will be found that

$$P - \xi Q = \frac{\xi^{2n+1} - 1}{\xi - 1}, \quad P + \xi Q = \frac{\xi^{2n+1} + 1}{\xi + 1};$$

so that the factors $(P \mp \xi Q)$ are simply the well known algebraic factors $(\xi^{2n+1} \mp 1) \div (\xi \mp 1)$.

9845. (Professor BORDAGE.)—Construct a triangle right-angled at A, knowing (1) the ratio of the bisectors of the angles B and C, limited at their point of intersection O; (2) the surface m.

Solution by H. J. WOODALL, A.R.C.S.

Let the two bisectors intersect at O; then

$$AO : BO = \sin ABO : \sin BAO = \sin \tfrac{1}{2}B : \sin \tfrac{1}{2}A$$

$$= \sin \tfrac{1}{2}(\tfrac{1}{2}\pi - A) : \sin \tfrac{1}{2}A = \cot \tfrac{1}{2}A - 1 : 2^{\frac{1}{2}} = k : 1 \quad \text{say};$$

therefore $$\cot \tfrac{1}{2}A = 1 + k - 2^{\frac{1}{2}},$$

whence we can find A; then a, b are given, because

$$2m = ab = b^2 \tan A;$$

therefore $$b = (2m \cot A)^{\frac{1}{2}}, \quad a = (2m \tan A)^{\frac{1}{2}}.$$

13987. (Professor SANJÁNA, M.A.)—The incircle of the triangle ABC touches the sides in X, Y, Z; θ, θ_1, θ_2, θ_3 are the BROCARD angles of the triangles XYZ, AYZ, BZX, CXY, respectively. Prove that

$$\cot\theta_1 + \cot\theta_2 + \cot\theta_3 - \cot\theta = \cot\omega + \cot\theta = \operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C.$$

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If XYZ be the excircle corresponding to BC, these formulæ take the form

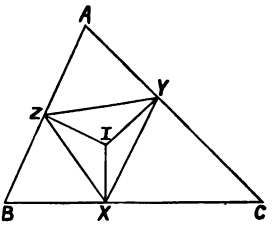
$$-\cot\theta_1+\cot\theta_2+\cot\theta_3-\cot\theta = -\cot\omega+\cot\theta$$
$$= -\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C;$$

so also for the other excircles.

Solution by G. N. Bates, B.A.; Professor A. Droz-Farny; *and others.*

(1) We have

$$\cot\theta_1 = \cot A+2\tan\tfrac{1}{2}A;$$



therefore

$$\Sigma\cot\theta_1 = \Sigma\cot A+2\Sigma\tan\tfrac{1}{2}A,$$
$$\cot\theta = \Sigma\tan\tfrac{1}{2}A;$$

therefore

$$\Sigma\cot\theta_1 = \Sigma\cot A+2\cot\theta;$$

therefore

$$\cot\theta_1+\cot\theta_2+\cot\theta_3-\cot\theta = \cot\omega+\cot\theta$$
$$= \cot\omega+\Sigma\tan\tfrac{1}{2}A = \operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C.$$

(2) In this case,

$$\cot\theta_1 = \cot A+2\tan\tfrac{1}{2}A, \qquad \cot\theta_2 = -\cot B+2\cot\tfrac{1}{2}B,$$
$$\cot\theta_3 = -\cot C+2\cot\tfrac{1}{2}C, \qquad \cot\theta = -\tan\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C;$$

therefore

$$-\cot\theta_1+\cot\theta_2+\cot\theta_3 = -\Sigma\cot A+2\cot\tfrac{1}{2}B+2\cot\tfrac{1}{2}C-2\tan\tfrac{1}{2}A;$$

therefore

$$-\cot\theta_1+\cot\theta_2+\cot\theta_3-\cot\theta = -\cot\omega+\cot\theta$$
$$= -\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C.$$

13554. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Sum to n terms the series $5+22+57+116+205+\dots$.

Solution by L. E. Reay, B.A.; Rev. T. Roach, M.A.; *and many others.*

$$5+22+57+116+205+\dots$$
$$= 1.5+2.11+3.19+4.29+5.41+\dots$$
$$= 1(2^2+1)+2(3^2+2)+3(4^2+3)+4(5^2+4)+5(6^2+5)+\dots;$$

therefore nth term is $n\{(n+1)^2+n\} = n(n^2+3n+1)$;

therefore sum to n terms is $\Sigma n^3+3\Sigma n^2+\Sigma n$

$$= \tfrac{1}{4}\{n^2(n+1)^2\}+\tfrac{1}{2}\{n(n+1)(2n+1)\}+\tfrac{1}{2}\{n(n+1)\}$$
$$= \tfrac{1}{2}\{n(n+1)\}\left[\tfrac{1}{2}\{n(n+1)\}+2n+1+1\right] = \tfrac{1}{2}\{n(n+1)^2\}(\tfrac{1}{2}n+2)$$
$$= \tfrac{1}{4}\{n(n+1)^2(n+4)\}.$$

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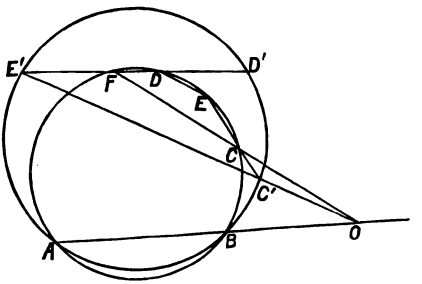
14061. (W. C. Stanham, B.A.)—Circles C_1, C_2 intersect at A and B. DE is a chord of C_1, C'E' a chord of C_2. DE' meets C_2 again at D', C'E meets C_1 again at C. If DE and C'E' are parallel, the anharmonic ratios (ABCD) and (ABC'D') are equal. Conversely, if (ABCD) and (ABC'D') are equal, DE and C'E' are parallel.

Solution by the Proposer.



Let DD' meet C_1 again at F, and let FC intersect C'E' at O. We have

$$\angle FE'C' = D'DE = FCE = C'CO;$$

therefore F, C, C', E' are concylic; therefore

$$OC.OF = OC'.OE';$$

therefore O is a point on AB. Therefore the intersections of corresponding rays of the pencils E' (ABC'D') and F (ABCD), which have a common ray E'F, are collinear; therefore (ABCD) = (ABC'D'). Conversely, if the anharmonic ratios are equal, then, since E' (ABC'D') and F (ABCD) have a common ray, O lies on AB; therefore $\angle FE'C' = D'DE$; therefore DE and C'E' are parallel.

14047. (Professor Umes Chandra Ghosh.)—Show that the volume generated by the revolution of a quadrant of an ellipse about a line through its centre whose inclination to the major axis is α, and which is outside the given quadrant, is

$$\tfrac{1}{3}\pi\left\{\frac{e^4\sin^2\alpha\cos^2\alpha - 1 + e^2}{(1-e^2\sin^2\alpha)^2}(a^3\cos^3\alpha + b^3\sin^3\alpha)\right.$$
$$+\frac{3b^2}{1-e^2\sin^2\alpha}(a\cos\alpha + b\sin\alpha) + \frac{2e^2\sin\alpha\cos\alpha}{(1-e^2\sin^2\alpha)^2}\frac{b}{a}(a^3\cos^3\alpha - b^3\sin^3\alpha)$$
$$\left.-\sin\alpha\cos\alpha(a^3\sin\alpha + b^3\cos\alpha)\right\}.$$

Solutions (1) *by* H. J. Woodall, A.R.C.S.; (2) *by* G. Birtwistle.

(1) By Guldin's Rule I., the volume required is equal to product of plane area multiplied by length of path of centroid. Plane area $= \frac{1}{4}\pi ab$, distance of centroid C from centre (of ellipse) $O = 4(a^2+b^2)^{\frac{1}{2}}/3\pi$, distance of centroid from the axis $OK = 4(a^2+b^2)^{\frac{1}{2}}/3\pi \times \cos\beta$, where

$$\beta = \angle COK, \quad \cos\beta = (a\sin\alpha + b\cos\alpha)/(a^2+b^2)^{\frac{1}{2}}.$$

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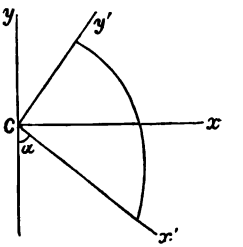
Hence path of centroid

$= 2\pi \times 4\,(a^2+b^2)^{\frac{1}{2}}/3\pi \times (a\sin\alpha + b\cos\alpha)/(a^2+b^2)^{\frac{1}{2}} = \frac{8}{3}\,(a\sin\alpha + b\cos\alpha)$.

Hence volume required $= \frac{4}{3}\pi ab\,(a\sin\alpha + b\cos\alpha)$.

(2) By GULDIN's theorem, volume generated $= 2\pi\bar{x}$ (area of quadrant), where $\bar{x}$ = distance of centre of gravity of area from axis of revolution. Now



$$\bar{x}' = 4a/3\pi, \quad \bar{y}' = 4b/3\pi,$$

and $$\bar{x} = \bar{x}'\sin\alpha + \bar{y}'\cos\alpha$$

$$= \tfrac{4}{3}\pi\,(a\sin\alpha + b\cos\alpha)\,;$$

$\therefore$ volume $= 2\pi \cdot \frac{1}{2}\pi ab \cdot \frac{4}{3}\pi\,(a\sin\alpha + b\cos\alpha)$

$$= \tfrac{4}{3}\pi ab\,(a\sin\alpha + b\cos\alpha).$$

[It appears as if the long expression given in the Question ought to reduce to the above simple form.]

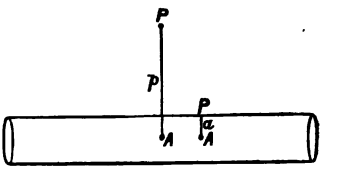
14003. (R. CHARTRES.)—The attraction of an infinitely long cylindrical bar on an external point according to the law of the inverse square of the distance is $2\gamma\rho\kappa/p$ where κ = area of a section of the bar and p = the distance of the external point from the bar, γ and ρ being the usual constants. Explain the absurdity arising from this when the point is on the cylinder, and deduce the real value.

Solution by the PROPOSER.

The infinite value is caused by not making p the distance to the centre of the bar.

Let a = radius of the section; therefore attraction

$$= \frac{2\gamma\rho\kappa}{p} = \frac{2\gamma\rho\pi a^2}{a} = 2\gamma\rho\pi a.$$



13934. (B. H. STEDE, M.A.)—Show that it follows immediately, by inversion, that any circle passing through the two points of intersection of one of the four axes of similitude and the orthogonal circle to three given circles cuts these circles at equal angles, and hence describe circles cutting three given circles at a given angle.

Solution by the PROPOSER.

Let an axis of similitude meet the orthogonal circle in the points L and M. Invert from one of these points, say L. The three given circles

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invert into three circles whose centres lie on the right line, which is the inverse of the orthogonal circle. To the point M, which is the intersection of the orthogonal circle and a right line (viz., an axis of similitude), cutting the three given circles at equal angles, will correspond on inversion the external centre of similitude of the inverse circles, and therefore to any circle through L and M will correspond on inversion any right line through this external centre of similitude; and, since this line cuts the inverse circles at equal angles, the circle through L and M will cut the original circles at equal angles.

The problem, to describe a circle cutting three given circles at a given angle, is therefore reduced to that of describing a circle passing through two given points and cutting a given circle at a given angle, which admits of a simple geometrical solution.

As we have four pairs of points L and M, we see how the eight required circles determine four coaxal systems.

14012. (Professor G. B. MATHEWS, M.A.)—Prove the identities—

(1) $a^5(b-c)+b^5(c-a)+c^5(a-b)+a(b-c)^5+b(c-a)^5+c(a-b)^5$
$+10abc(b-c)(c-a)(a-b)=0;$

(2) $a^7(b-c)+b^7(c-a)+c^7(a-b)+a(b-c)^7+b(c-a)^7+c(a-b)^7$
$+7abc(b-c)(c-a)(a-b)\{3(a^2+b^2+c^2)-2(bc+ca+ab)\}=0.$

Solution by LIONEL E. REAY, B.A.; G. W. PRESTON, B.A.; *and many others.*

(1) $a^5(b-c)+b^5(c-a)+c^5(a-b)+a(b-c)^5+b(c-a)^5+c(a-b)^5$

has $(b-c)$ a factor, since, on putting $b=c$, the expression reduces to

$$c^5(c-a)+c^5(a-c)+c(c-a)^5+c(a-c)^5=0.$$

Similarly, $(c-a)$, $(a-b)$ are factors.

Putting $a=0$, the expression again vanishes; therefore abc is a factor.

Since the expression is of the 6th degree and skew-symmetrical, N is equal to $pabc(b-c)(c-a)(a-b)$, where p is numerical and independent of a, b, and c.

Consider the coefficients of the term ab^3c^2; we get $-p=10$.

Hence the theorem follows.

(2) As in (1), $abc(b-c)(c-a)(a-b)$ is a factor; therefore

$$\Sigma a^7(b-c)+\Sigma a(b-c)^7$$
$$\equiv abc(b-c)(c-a)(a-b)\{p(a^2+b^2+c^2)+q(bc+ca+ab)\},$$

since the remaining factor must be symmetrical and of the second degree.

Consider the term a^5b^2c, $p=-21$.

Consider the term a^4b^3c, $p-q=-35$; therefore $q=+14$.

Hence we get $-7abc(b-c)(c-a)(a-b)\{3(\Sigma a^2)-x(\Sigma bc)\}$.

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13778. (Professor CROFTON, F.R.S.)—If Δ is a triangular number, $9\Delta+1$ is also one; also all triangular numbers, not multiples of 3, are of the form $9\Delta+1$. If Δ, Δ' are any two triangular numbers, $4(\Delta+\Delta')+1$ is the sum of two squares. If $4n+1$ is a prime number, $n=\Delta+\Delta'$.

Solution by Professor A. DROZ-FARNY; R. W. D. CHRISTIE; *and others.*

Posons $$\Delta=\frac{K(K+1)}{2};$$

$$9\Delta+1=\frac{9K(K+1)+2}{2}=\frac{(3K+1)(3K+2)}{2}.$$

La réciproque est évidente.

De même

$$4(\Delta+\Delta')+1=\frac{4K(K+1)+4K'(K'+1)+2}{2}=(K-K')^2+(K+K'+1)^2.$$

Tout nombre premier de la forme $4n+1$ se laisse d'après le théorème d'EULER, mettre d'une seule manière sous la forme A^2+B^2, A et B étant de parité différente. On pourra donc toujours poser

$$A=K-K',\quad B=K+K'+1,$$

K et K′ seront exprimés en nombres entiers, et par conséquent

$$n=\Delta+\Delta'.$$

14063. (ROBERT W. D. CHRISTIE.)—The sum of any $\frac{1}{2}(p-1)$ *consecutive odd* squares, *plus* a triangular, may be resolved into the same number of *consecutive even* squares, *plus* a triangular, when the triangulars are non-consecutive, if $p>3$. *Ex. gr.*,

$$3^2+5^2+7^2+1=2^2+4^2+6^2+28.$$

[This theorem has no connexion with Dr. GLAISHER's similar theorem, viz., that a square *plus* a triangular may be resolved into another square *plus* a triangular.]

Solution by the PROPOSER.

If we take any odd number of consecutive triangulars

$$T_n+T_{n+1}+T_{n+2}\ldots T_{2n-1},$$

and add them, the sum will be the same in whatever way we add them.

Thus $$T_n+(T_{n+1}+T_{n+2})+(T_{n+3}+T_{n+4})+\ldots.$$
$$\equiv T_{2n-1}+(T_n+T_{n+1})+(T_{n+2}+T_{n+3})+\ldots$$
$$=T_n+A^2+B^2+\ldots=T_{2n-1}+C^2+D^2+\ldots,$$

since $$T_m+T_{m+1}=(m+1)^2.$$

Here the triangulars are not consecutive.

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13960. (Professor UMES CHANDRA GHOSH).—Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{2n+1}\phi}{(1+\sin^2\phi)^{n+1}}\,\delta\phi = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left[\frac{1}{2}(n+1)\right]}{\Gamma\left[\frac{1}{2}(n+2)\right]}.$$

Solution by H. W. CURJEL, M.A.; *and* F. H. PEACHELL, B.A.

Let $$\frac{\cos^2\phi}{1+\sin^2\phi} = x,$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{2n+1}\phi}{(1+\sin^2\phi)^{n+1}} = \tfrac{1}{2}\int_0^1 \frac{x^n}{\sqrt{(1-x^2)}}\,dx = \tfrac{1}{4}\int_0^1 y^{\frac{1}{2}(n-1)}(1-y)^{-\frac{1}{2}}\,dy,$$

(where $x^2 = y$)

$$= \tfrac{1}{4}\frac{\Gamma\left[\frac{1}{2}(n+1)\right]\Gamma\left(\frac{1}{2}\right)}{\Gamma\left[\frac{1}{2}(n+2)\right]} = \tfrac{1}{4}\sqrt{\pi}\,\frac{\Gamma\left[\frac{1}{2}(n+1)\right]}{\Gamma\left[\frac{1}{2}(n+2)\right]}.$$

[Professor NANSON solves the Question as follows:—"Let $\sin\phi = \tan\frac{1}{2}\theta$, and the integral becomes $\frac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^n\theta\,d\theta$, which has the value given."]

10500. (D. BIDDLE.)—Prove that

$$\pi = \left(366\tfrac{1}{2} + \Sigma_1^\infty \frac{9}{n^2(n+1)^2\left\{3+16n(n+1)\right\}}\right)^{\frac{1}{2}} - 16.$$

[The materials for solving this Question are contained in Vol. L., App. I. The formula is a convenient one for finding π.]

Solution by the PROPOSER.

$$\tfrac{1}{4}\pi = \tan^{-1}1 = 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots = \tfrac{2}{3}+\tfrac{2}{3\cdot5}+\tfrac{2}{5\cdot9}+\ldots$$

$$= \frac{2}{3}+\frac{2}{3+32}+\frac{2}{3+3\cdot32}+\frac{2}{3+6\cdot32}+\frac{2}{3+10\cdot32}+\frac{2}{3+15\cdot32}+\ldots.$$

Let $a = \frac{3}{32}$, $k_1 = 1$, $k_2 = 1+2 = 3$, $k_3 = 1+2+3 = 6$, ...

Then $$4\pi = \frac{1}{a}+\frac{1}{k_1+a}+\frac{1}{k_2+a}+\frac{1}{k_3+a}+\ldots = \frac{1}{a}+\Sigma_1^\infty\frac{1}{k_n+a}$$

$$= \frac{1}{a}+\Sigma_1^\infty\left(\frac{1}{k_n}-\frac{a}{k_n^2}+\frac{a^2}{k_n^2(k_n+a)}\right)$$

$$= \frac{1}{a}+\Sigma_1^\infty\left\{\frac{2}{n(n+1)}-\frac{4a}{n^2(n+1)^2}+\frac{8a^2}{n^2(n+1)^2\left\{n(n+1)+2a\right\}}\right\}$$

$$= 10\tfrac{2}{3}+2-\tfrac{3}{32}(\tfrac{1}{6}\pi^2-12)+\Sigma_1^\infty\frac{9}{8n^2(n+1)^2\left\{3+16n(n+1)\right\}};$$

therefore $$\pi^2+32\pi = 110\tfrac{1}{3}+\Sigma_1^\infty\frac{9}{n^2(n+1)^2\left\{3+16n(n+1)\right\}},$$

and the result in the Question follows.

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$\Sigma_1^\infty \frac{9}{n^2(n+1)^2\{3+16n(n+1)\}}$, carried to 11 places of decimals, = ·06723598263, and this gives π to 14 places of decimals, viz., 3·14159265358979....

Subjoined are the values of $n^2(n+1)^2\{3+16n(n+1)\}$ from $n=1$ to $n=40$. The reciprocals of these values, after the first ten, are easily found by the aid of tables.

n.	$n^2(n+1)^2\{3+16n(n+1)\}$.	n.	$n^2(n+1)^2\{3+16n(n+1)\}$.
1.	140.	21.	1578418380.
2.	3564.	22.	2073635564.
3.	28080.	23.	2692059840.
4.	129200.	24.	3457080000.
5.	434700.	25.	4395267500.
6.	1190700.	26.	5536652940.
7.	2819264.	27.	6915014064.
8.	5987520.	28.	8568175280.
9.	11688300.	29.	10538318700.
10.	21332300.	30.	12872306700.
11.	36851760.	31.	15622016000.
12.	60815664.	32.	18844683264.
13.	96556460.	33.	22603262220.
14.	148308300.	34.	26966792300.
15.	221356800.	35.	32010778800.
16.	322200320.	36.	37817584560.
17.	458722764.	37.	44476833164.
18.	640377900.	38.	52085823660.
19.	878385200.	39.	60749956800.
20.	1185937200.	40.	70583172800.

14060. (R. KNOWLES, B.A.)—Tangents are drawn from a point T (hk), to meet the ellipse $a^2y^2+b^2x^2=a^2b^2$, centre C, in P and Q; prove (1) that

$$CP^2+CQ^2=\frac{2(a^2k^2+b^2h^2-a^2b^2)(a^4k^2+b^4h^2)+2a^4b^4(h^2+k^2)}{(a^2k^2+b^2h^2)^2};$$

(2) if CP, CQ are conjugate, the locus of T is the ellipse

$$a^2y^2+b^2x^2=2a^2b^2.$$

Solution by R. TUCKER, M.A.; *and* Rev. T. MITCHESON, B.A.

Polar of T is $b^2hx+a^2ky=a^2b^2$ ..... (1).

Let P, Q be ϕ, ϕ'; then (1) becomes

$$bh\cos\phi+ak\sin\phi=ab;$$

whence, if $\cos\phi\equiv z$, $\sin\phi=z'$, $a^2k^2+b^2h^2\equiv\lambda$,

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we get $\lambda z^2 - 2ab^2hz + a^2(b^2 - k^2) = 0,$

$\lambda z'^2 - 2a^2bkz' + b^2(a^2 - h^2) = 0.$

Therefore $CP^2 + CQ^2 = a^2(\cos^2\phi + \cos^2\phi') + b^2(\sin^2\phi + \sin^2\phi')$

$= \left[2a^4\left\{b^4h^2 + k^2(\lambda - a^2b^2)\right\}\right.$

$\left. + 2b^4\left\{a^4k^2 + h^2(\lambda - a^2b^2)\right\}\right]/\lambda^2$

$= \left\{2a^4b^4(h^2 + k^2) + 2(a^4k^2 + b^4h^2)(\lambda - a^2b^2)\right\}/\lambda^2.$

For conjugate diameters, CQTP is a parallelogram; therefore, since $CK.CT = CR^2$ (if CT cuts the curve in R), we have

$$CT^2 = 2CR^2;$$

therefore locus is the ellipse $a^2y^2 + b^2x^2 = 2a^2b^2$.

13958. (Professor SANJÁNA, M.A.)—Resolve into prime factors:—

(1) $134^4 + 3^4$, (2) $86^4 + 1$, (3) $1786^4 + 9^4$, (4) $3266^4 + 1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

$N_1 = 134^4 + 3^4$. Here -1, 2, 3 are quadratic residues, so that every divisor is of form $(24f + 1)$. And, trying prime divisors of this form, it is found that N has no divisor < 1000.

Ex. gr., Let $p = 577$. Then $134^2 \equiv 69$ and $134 \equiv 145 \pmod{577}$; therefore 577 is a *non*-divisor.

Next, applying the method of factorization given in Mr. F. W. LAURENCE's paper "On Determination of certain Primes," in *Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 465, it is found that

$$N_1 = 16729 \cdot 19273.$$

[It is almost impossible to give the details of this factorization in any short compass; it would involve explanation of LAURENCE's method at some length, or else reading along with his paper.]

$N_2 = 86^4 + 1^4$. Here every divisor is of form $(8f + 1)$. REUSCHLE's *Tafeln complexer Primzahlen*, Berlin, 1875, p. 443, show that there is no divisor < 1000. Applying LAURENCE's method as before,

$$N_2 = 7129 \cdot 7673.$$

$N_3 = 1786^4 + 9^4$. Here every divisor is of form $(8f + 1)$; by same process as with N_1, it is found that 17 is a divisor and that there are no more divisors < 1000.

$N_4 = 3266^4 + 1^4$. Here every divisor is of form $(8f + 1)$. REUSCHLE's Tables give 17 and 41 as divisors, and show that there are no more divisors < 1000.

[Some further information is required to enable the factorization of N_3, N_4 to be completed.]

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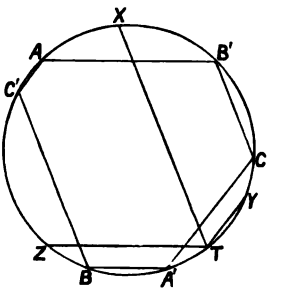
14100. (Professor ELLIOTT, F.R.S.)—ABC, A′B′C′ are two triangles inscribed in a conic, and PQR is the PASCAL line of the hexagon AB′CA′BC′, P being the intersection of B′C and BC′, Q of C′A and CA′, and R of A′B and AB′. Prove that, if XYZ be an inscribed triangle which is copolar both with ABC and A′B′C′, with vertices homologous in the order named, then XP, YQ, ZR intersect on the conic, and the two poles lie on the line PQR. Prove also that all inscribed triangles XYZ which are copolar both with ABC and A′B′C′ can be constructed by joining P, Q, R to any the same point of the conic, and taking for vertices the second intersections of the conic with the joining lines.

Solution by Professor A. DROZ-FARNY.

Soient ω le centre d'homologie des triangles ABC et XYZ, et ω' celui des triangles A′B′C′ et XYZ. Les hexagones inscrits BYB′CZC′, AXA′CZC′, AXA′BYB′ admettant respectivement les Pascales, $\omega\omega'$P, $\omega\omega'$Q, $\omega\omega'$R, il en résulte que les cinq points $\omega\omega'$PQR sont en ligne droite.

Tirons les droites XP, YQ, ZR et supposons que XP et YQ se coupent en α; XP et ZR se coupent en α'.

Or dans l'hexagone XαYB′CA′, Xα et B′C se coupent en P, αY et CA′ se coupent en Q, YB′ et A′X se coupent en ω'; ces trois points étant en ligne droite α appartient à la conique XYB′CA′. De même dans l'hexagone Xα'ZC′BA′, Xα' et C′B se coupent en P, α'Z et BA′ se coupent en R, ZC′ et A′X se coupent en ω', et comme ces trois points sont en ligne droite α' appartient a la même conique. Or les deux points α et α' doivent coïncider comme constituant les deux la même seconde intersection du rayon XP avec la conique. La proposition réciproque se démontre d'une manière analogue.



[Mr. CURJEL solves the Question as follows:—Project the points in which PQR cuts the conic into the circules; the conic becomes a circle and the triangles ABC, A′B′C′ become equal in all respects. The chords AA′, BB′, CC′ are equal; therefore angles AXA′, BYB′, CZC′ are equal or supplementary; therefore the poles O, O′ are at infinity, since they cannot be on the circle XYZ; *i.e.*, O, O′ are on PQR. Let XP (*i.e.*, parallel to BC′ through X) cut the circle in T. Then arc AY = arc XB = arc C′T; therefore YT is parallel to AC′, *i.e.*, passes through Q; similarly ZT passes through R. Again, if TX, TY, TZ are parallel to BC′, CA′, AB′, arc BX = arc C′T = arc EY; therefore AX and BY are parallel; similarly CZ is parallel to both. Hence the required results.]

14121. (C. N. CAMA.)—If two parabolas touch at P, and their axes meet at O, the intersections of the focal distance to P of one with the

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axis of the other lie on a circle which passes through P and O. Hence, if S, S′ be fixed points on fixed lines meeting at right angles in O, and two systems of parabolas be drawn having S, S′ for their foci, and axes OS, OS′, such that each parabola of one system touches the corresponding parabola in the other system, all the points of contact lie on a fixed circle. Generally, four parabolas of one system can be drawn to touch a given parabola of the other system. If the axes be parallel, show that the corresponding locus is a straight line.

Solution by the PROPOSER.

Let S, S′ be the foci, P the point of contact, SU, S′U′ the axes, T, T′ the intersections of SP, S′P with the respective axes. Then

$$\angle SPU = OUU', \quad S'PU = OU'U;$$

therefore $\angle SPT = S'OU$; therefore O, P, T, T′ are concyclic (Fig. 1).

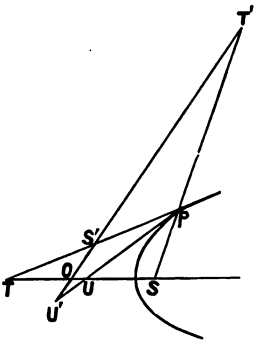

Fig. 1.

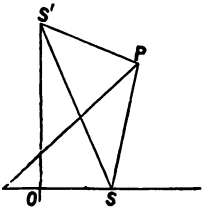

Fig. 2.

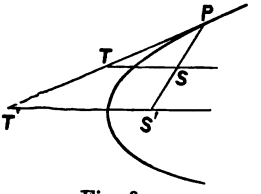

Fig. 3.

If the $\angle SOS'$ be a right angle, it follows that $\angle SPS'$ is so also; therefore P lies on a fixed circle, viz., the one with SS′ as diameter (Fig. 2).

Also, if the axes are parallel, it will be seen that P lies on SS′, which is therefore its locus (Fig. 3).

Now, if a particular parabola with focus S be drawn, and the circle round SOS′ cut it in P, it will generally cut the parabola in three other points which will therefore be the points of contact of three other parabolas of the second system.

14058. (Rev. T. MITCHESON, B.A.)—If (h, k) be a point external to the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$, and θ the angle at which a tangent from the point is inclined to the major axis, then

$$2hk = (h^2 - a^2)\tan\theta + (k^2 - b^2)\cot\theta.$$

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Solution by G. BIRTWISTLE.

A line through (h, k) making an angle θ with major axis of ellipse is

$$(y-k) = \tan\theta\,(x-h) \quad \text{or} \quad y - x\tan\theta = k - h\tan\theta.$$

This touches the ellipse, $x^2/a^2 + y^2/b^2 = 1$, if

$$a^2(-\tan\theta)^2 + b^2(1)^2 = (k - h\tan\theta)^2,$$

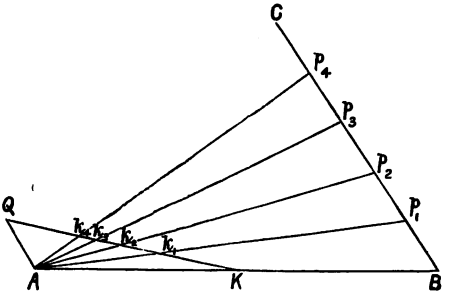
$$\text{or} \quad (h^2-a^2)\tan^2\theta - 2kh\tan\theta + (k^2-b^2) = 0,$$

$$\text{or} \quad 2kh = (h^2-a^2)\tan\theta + (k^2-b^2)\cot\theta.$$

14017. (D. BIDDLE.)—In the straight line AB a point K is taken, such that

$$AK : KB = 1 : n,$$

and another straight line BC, forming any angle with AB, is divided into r equal parts by points $p_1, p_2, \ldots$. The parallelogram ABp_1Q being completed, and the necessary joins being effected, KQ intersects Ap_1, Ap_2, ... in $k_1, k_2, \ldots$. Prove that



$$Ak_1 : k_1p_1 = 1 : n+1, \quad Ak_2 : k_2p_2 = 1 : n+2, \quad \ldots,$$

and

$$Ak_r : k_rp_r = 1 : n+r.$$

[Of course, parallels to BC from $k_1, k_2, \ldots$ divide AB in corresponding ratios. But it would be interesting if the above theorem could be utilized for the summation of the reciprocals of successive numbers.]

Solution by H. W. CURJEL, M.A., *and many others.*

Draw $Bq_1q_2\ldots q_r$ parallel to KQ, cutting $Ap_1Ap_2\ldots Ap_r$ in $q_1, q_2, \ldots q_r$.

Then

$$Ak_r : k_rq_r = 1 : n,$$

and $\triangle Bq_rp_r$ is similar to $\triangle Qk_rA$; therefore

$$q_rp_r = r \,.\, Ak_r; \quad \text{therefore} \quad Ak_r : k_rp_r = 1 : n+r.$$

[As to the summation of the reciprocals of successive numbers, it is easy to see that a line representing the proportional value of such sum could be determined by a series of bisections of Kk_1, k_2k_3, k_4k_5, &c., then projecting the mid-points to double their distance from A, and repeating the process. But, unless the number of terms were 2^n, other aliquot parts would have to be taken at the finish.]

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14067. (Professor E. Lemoine.)—Soit M un point dont les coordonnées normales sont x, y, z par rapport à un triangle de référence ABC; AM, BM, CM coupent BC, CA, AB en A′, B′, C′; A_1, B_1, C_1 sont les milieux des hauteurs du triangle ABC. Trouver le lieu de M si $A'A_1$, $B'B_1$, $C'C_1$ sont concourantes en un point N.

Solutions (1) *by the* Proposer; (2) *by* R. Tucker, M.A.

(1) Le lieu est la cubique $\Sigma yz(b^2y\cos C - c^2z\cos B) = 0$.

Remarques.—La cubique passe par les trois sommets.

Elle passe par le point $1/(a\tan A)$, $1/(b\tan B)$, $1/(c\tan C)$, par l'orthocentre et par le point de Nagel $(p-a)/a$, $(p-b)/b$, $(p-c)/c$, et par ses transformés continus en A, B, C. Les points N sont respectivement le barycentre, l'orthocentre, le centre du cercle inscrit et ceux des cercles exinscrits. Le lieu des points N est la cubique $\Sigma ayz(by-cz) = 0$.

Elle coupe les côtés aux points où les coupent les droites qui joignent les sommets au point $1/(a\tan A)$, ...

(2) A_1 is 1, cos C, cos B, and A′ is 0, y, z; therefore equation to A_1A is

$$(z\cos C - y\cos B)\alpha - z\beta + y\gamma = 0.$$

Hence locus is

$$\begin{vmatrix} z\cos C - y\cos B & -z & y \\ z & x\cos A - z\cos C & -x \\ -y & x & y\cos B - x\cos A \end{vmatrix} = 0;$$

i.e., the cubic $\Sigma a^2x^2(y\cos B - z\cos C) = 0$.

The locus therefore passes through A, B, C, the centroid, and the orthocentre.

14071. (Professor G. B. Mathews, F.R.S.)—Prove that the envelope of the line

$$\lambda\xi(\mu\eta^2 - \nu\zeta^2)x + \mu\eta(\nu\zeta^2 - \lambda\xi^2)y + \nu\zeta(\lambda\xi^2 - \mu\eta^2)z = 0,$$

subject to the condition $\eta\zeta + \zeta\xi + \xi\eta = 0$,

is, in general, a curve of the third class and fourth order; and investigate the relations connecting λ, μ, ν in any exceptional cases.

Show how this analytical problem is connected with the following:—The sides BC, CA, AB of a triangle inscribed in a conic meet a fixed line u in the points α, β, γ; P is any point on the conic, and $P\alpha$, $P\beta$, $P\gamma$ meet the conic again in A′, B′, C′. It may be proved that AA′, BB′, CC′ meet in a point Q on u; it is required to find the envelope of the line PQ as P moves round the conic. [A special case is the envelope of the axis of a parabola which touches three given lines.]

Solution by Rev. J. Cullen.

If we eliminate η (say) between the equation of the given line and the condition $\eta\zeta + \xi\zeta + \eta\xi = 0$ and put $\zeta = \rho\xi$, we get

$$\rho^3L + \rho^2M + \rho N + K = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

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where L, &c., are of the first degree in x, y, and z. Hence, since this line is a tangent to the curve, it is clear that in general only three tangents can be drawn from any assigned point. The curve is therefore of the third class. Also, since the discriminant of a cubic is of the fourth degree in the coefficients, it is also of the fourth order. λ, μ, and ν are included in L, &c., so that in any exceptional case the conditions can be easily determined. Taking ABC as the triangle of reference, the circumscribing conic is $lyz + mxz + nxy = 0$; let u be $l'x + m'y + n'z = 0$ and P be $x/x' = \ldots = \ldots$. Then, in the ordinary manner, we find Q to be

$$xx'l'/l = \ldots = \ldots .$$

Therefore the line PQ is

$$x'l'(y'^2m'n - z^2mn')x + \ldots + \ldots = 0;$$

and putting $x' = l\xi$, &c., we see that the envelope of this line is of the same form as that of (1), differing only with regard to its constants.

14014. (Professor Sanjána, M.A.)—In the triangle ABC, AD is the median from A; AE and AE′ are the internal and external bisectors of the angle A. Prove that

$$AD \cos DAE = \tfrac{1}{2}(AB + AC) \cos \tfrac{1}{2}A,$$

and

$$AD \cos DAE' = \tfrac{1}{2}(AB \sim AC) \sin \tfrac{1}{2}A.$$

Also apply these equalities to prove Feuerbach's theorem.

Solution by E. P. Barrett, B.A.; *and* L. E. Reay, B.A.

In figure AN = AC, N being in BA produced; ADA′ = 2AD, A′P, B$ma'pn$, CaM, AE′ perpendicular to AE, Mm, A′a', Nn parallel to AE. Then BA′ is equal and parallel to AC, therefore

$$Ba' = aC = pn;$$

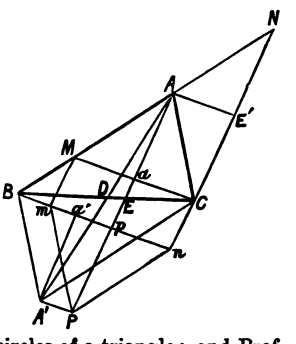
therefore Pn is equal and parallel to AN, Pm equal and parallel to A′B; therefore

$$Nn = AP \quad \text{and} \quad Bm = A'P,$$

or $(AB + AC) \cos \tfrac{1}{2}A = 2AD \cos DAE$

and $(AB \sim AC) \sin \tfrac{1}{2}A = 2AD \sin DAE$

$= 2AD \cos DAE'$.



[Feuerbach's theorem, referred to in the Question, is that the nine-point circle touches the inscribed and escribed circles of a triangle; and Prof. U. C. Ghosh solves this part of the Question as follows:—In the figure given on p. 105 of Casey's *Sequel* (4th edition), let Σ be the centre of the nine-points circle, and draw $\Sigma N'$ parallel to ON. Then

$$ON = AN - AO = AF \cos\theta - r \operatorname{cosec} \tfrac{1}{2}A,$$

where $\theta = \angle FAN$, and $\Sigma N' = \tfrac{1}{2}FH = \tfrac{1}{2}R \sin(\tfrac{1}{2}A + B)$;

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therefore $ON - \Sigma N' = AF\cos\theta - \frac{1}{2}R\sin(\frac{1}{2}A+B) - r\operatorname{cosec}\frac{1}{2}A$.
And $FN = AF\sin\theta$ and $FN' = \frac{1}{2}FG = \frac{1}{2}R\cos(\frac{1}{2}A+B)$;
therefore $NN' = AF\sin\theta - \frac{1}{2}R\cos(\frac{1}{2}A+B)$.
Therefore

$$\Sigma O^2 = AF^2 + \tfrac{1}{4}R^2 - AF.R\sin(\theta+\tfrac{1}{2}A+B) - 2AF\cos\theta.r\operatorname{cosec}\tfrac{1}{2}A + r.R\operatorname{cosec}\tfrac{1}{2}A\sin(\tfrac{1}{2}A+B) + r^2\operatorname{cosec}^2\tfrac{1}{2}A.$$

Now, since $\angle AF\Sigma = \frac{1}{2}A - (\theta+\frac{1}{2}A+B)$,

$$\Sigma A^2 = AF^2 + \tfrac{1}{4}R^2 - AF.R.\sin(\theta+\tfrac{1}{2}A+B) = \tfrac{1}{4}(R^2+b^2+c^2-a^2).$$

Therefore

$$\begin{aligned}\Sigma O^2 &= \tfrac{1}{4}(R^2+b^2+c^2-a^2) - \{(b+c)\cos\tfrac{1}{2}A - R\sin(\tfrac{1}{2}A+B)\}\, r\operatorname{cosec}\tfrac{1}{2}A + r^2\operatorname{cosec}^2\tfrac{1}{2}A\\ &= \tfrac{1}{4}R^2 + \tfrac{1}{4}(b^2+c^2-a^2) - (b+c)\cos\tfrac{1}{2}A.r/\sin\tfrac{1}{2}A + rR(b+c)/a + r^2/\sin^2\tfrac{1}{2}A\\ &= \tfrac{1}{4}R^2 + \tfrac{1}{4}(b^2+c^2-a^2) - Rr + Rr - (b+c)\cos\tfrac{1}{2}A.r\operatorname{cosec}\tfrac{1}{2}A + rR(b+c)/a + r^2\operatorname{cosec}^2\tfrac{1}{2}A\\ &= \tfrac{1}{4}R^2 - Rr + \tfrac{1}{4}(b^2+c^2-a^2) + Rr.2s/a - (b+c)r\cot\tfrac{1}{2}A + r^2\operatorname{cosec}^2\tfrac{1}{2}A\\ &= \tfrac{1}{4}R^2 - Rr + \{\tfrac{1}{4}(b^2+c^2-a^2) + \tfrac{1}{2}bc\} - (b+c)(s-a) + \{bc(s-a)\}/s\\ &= \tfrac{1}{4}R^2 - Rr + s(s-a) - (b+c)(s-a) + bc.(s-a)/s\\ &= \tfrac{1}{4}R^2 - Rr + \{(s-a)(s-b)(s-c)\}/s = \tfrac{1}{4}R^2 - Rr + r^2 = (\tfrac{1}{2}R-r)^2.\end{aligned}$$

In the same way the other cases may be proved.]

14023. (Rev. T. Roach, M.A.)—Point out the fallacy in the following statement:— $\tan^{-1}\dfrac{x}{1-x^2} + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{3x(1-x^2)}{1-4x^2+x^4}$.

Now, when $x = 1$, we have

$$\tan^{-1}\infty + \tan^{-1}\infty = \tan^{-1}0; \quad \text{therefore} \quad \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = 0.$$

Solution by Professor Umes Chandra Ghosh, M.A.; G. W. Preston, B.A.; *and many others.*

$\tan^{-1}\infty = n\pi + \frac{1}{2}\pi$ and *not* $\frac{1}{2}\pi$ *only*, and $\tan^{-1}0 = n'\pi$ and *not* 0 *only*, where n and n' are *any* integers. Hence

$$\tan^{-1}\infty + \tan^{-1}\infty = n\pi + \tfrac{1}{2}\pi + m\pi + \tfrac{1}{2}\pi = (n+m+1)\pi = \text{a multiple of } \pi = \tan^{-1}0.$$

[Mr. Bickmore observes that $\tan^{-1}y$ is an inverse symbol, and requires definition: "the least positive angle with tangent $y = \tan^{-1}y$." Then

$$\tan^{-1}y + \tan^{-1}z = \tan^{-1}\{(y+z)/(1-yz)\},$$

if $\tan^{-1}y$ and $\tan^{-1}z$ be each $< \frac{1}{2}\pi$. But in the case considered each is equal to a right angle, so that their sum is two right angles. Hence the true formula is, in this case,

$$\tan^{-1}y + \tan^{-1}z = \pi + \tan^{-1}\{(y+z)/(1-yz)\}.]$$

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13970. (C. E. Hillyer, M.A.)—F is a given point within an ellipse; CF meets the curve in P. Required to draw through F a straight line meeting the curve in Q, so that the angle between FQ and the tangent at Q is equal to that between CP and the tangent at P.

Solution by the Proposer.

In CF produced take E so that

$$CE \cdot CF = CA^2 + CB^2.$$

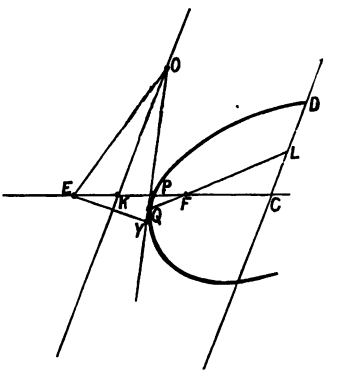
Let OK, the polar of F, meet CE in K. Take Y the image of E with respect to OK. Draw tangents YQ, YR from Y to the ellipse; then FQ, FR shall be the straight lines required.

For, if YQ meet OK in O, then QF is the polar of O.

Therefore (by Quest. 13359) QF and EO are antiparallel with respect to the angle between the conjugate diameters CP, CD, *i.e.*, if QF meet CD in L,

$$\angle FLC = OEK.$$

But, since CL is parallel to OK,



$$\angle FQO = QOK + FLC = EOK + OEK = OKC$$

= angle between CP and the tangent at P.

14022. (B. N. Cama, M.A.)—The pedal equation of a curve is given by $r^3 = pa^2$. In polars, this reduces to the lemniscate $r^2 = a^2 \sin 2\theta$. Can it be shown in a similar manner that the same equation may as well represent a fixed circle?

Solution by H. W. Curjel, M.A.

The equation $a^2p = r^3$ gives $a^4u^6 = u^2 + (du/d\theta)^2$, of which $r = a$ is a singular solution. This represents the envelope of the lemniscates $r^2 = a^2 \sin(2\theta + \alpha)$ represented by the general solution.

13966. (R. F. Davis, M.A.)—If B, C are given fixed points, and the angle BPC is constant, prove geometrically that the complete envelope of the inscribed circle of the triangle PBC is the base BC, and a parallel to a two-cusped epicycloid.

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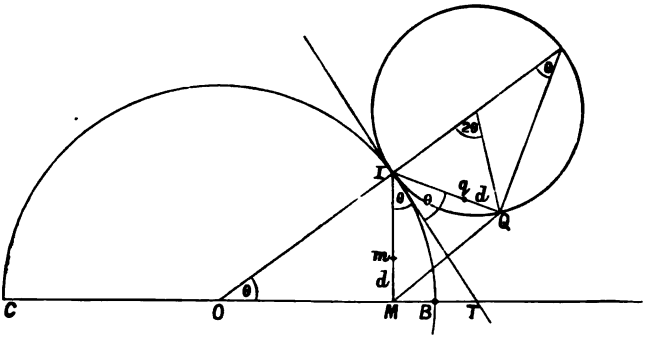
Solution by the PROPOSER.

The circumcircle of the triangle PBC is fixed; and, if E is the middle point of its arc BC, then, by a well-known property, I, the incentre, lies on PE and EI = EB = EC.

The question therefore may be stated thus: Find the envelope of a circle whose centre moves along the circumference of a fixed circle, and



which touches a given right line (WILLIAMSON's *Diff. Cal.*, Chap. XV., Ex. 15).

By considering the ultimate intersections of two consecutive circles of the series, the determination of the envelope is identical with the determination of a certain locus. Bearing in mind the fact that the common chord of two circles bisects common tangents and is perpendicular to the line of centres, we now require to solve the question:—If I be any point on the arc of a given circle cut off by a given chord BC, and IM perpendicular to BC, find the locus of the image of M in the tangent to the circle at I.

An examination of the particular case when BC is a diameter of the given circle easily discloses the locus required is the epicycloid formed by rolling upon the given circle a circle of half its dimensions.

When BC is no longer a diameter, but at a distance ($= d$) from the centre, the locus required is the curve formed by setting off a distance d along the normals to the epicycloid drawn internally.

13876. (B. H. STERDE, M.A.)—If r be radius vector to any point P on a curve, ρ the radius of curvature at P, r' and ρ' the corresponding values for corresponding point on inverse curve, and ρ'' the radius of curvature of reciprocal curve at point of contact of tangent corresponding to point P, k radius of circle of reciprocation, prove

$$\frac{r}{\rho} + \frac{r'}{\rho'} = \frac{2k^3}{(\rho\rho'')^{\frac{1}{2}}}.$$

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Solution by the PROPOSER; *by* Rev. J. CULLEN, B.A.; *and by* Professors NANSON *and* UMES CHANDRA GHOSH.

If p be the perpendicular from origin on tangent at P, p'' the perpendicular on tangent at corresponding point on reciprocal curve, r and r'' the corresponding radii vectores; ϕ the angle between radius vector and normal, it is easy to prove from the equations

$$\rho = \frac{rdr}{dp}, \quad \rho'' = \frac{r''dr''}{dp''}, \quad rp'' = r''p = k^2, \quad \frac{p}{r} = \frac{p''}{r''} = \cos\phi,$$

that $$\rho\rho'' \cos^3\phi = k^2,$$

a result given in WILLIAMSON'S *Diff. Calculus*.

Also $$\frac{r}{\rho} + \frac{r'}{\rho'} = 2\cos\phi.$$

Hence result follows.

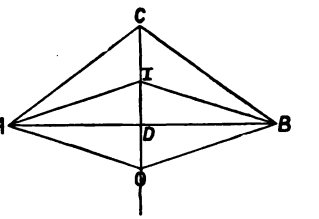
14045. (Professor SANJÁNA.)—In a triangle ABC both AO = AI and BO = BI, where O and I are the circum- and in-centre, respectively. Determine the angles.

Solution by H. J. WOODALL, A.R.C.S.; Professor T. SAVAGE; *and many others.*

(1) If O and I are on the same side of AB, they must be coincident; whence we get the equilateral triangle.

(2) If O and I are on opposite sides, then $\angle OAB = \angle BAI = \angle IAC$.

But OAC is an isosceles triangle, and is similar in all respects to OBC. Therefore the three angles BAC, ACB, CBA are as 2 : 6 : 2, and hence they are 36°, 108°, 36°, respectively.



[The PROPOSER remarks:—

As AIBO is a rhombus, $IO = 2ID = 2r$;

thus $$R^2 - 2Rr = 4r^2;$$

whence $$R = r(\sqrt{5}+1),$$

i.e., $$r/R = \sin IAD = 1/(\sqrt{5}+1) = \tfrac{1}{4}(\sqrt{5}-1).$$

Hence $$IAD = 18°.]$$

13571. (Rev. W. ALLEN WHITWORTH, M.A.)—A stick 18 inches long is divided at random into three parts. John is to have the longest, Tom the next, and Sam the shortest. Determine their expectations.

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Solution by the PROPOSER.

Let x denote the smallest, $x+y$ the next, and $x+y+z$ the largest piece; the whole stick being represented by unity. Then x, y, z are random quantities subject to the condition

$$3x+2y+z=1.$$

The expectations of the three terms in the first member of this equation must be equal. Therefore each $=\frac{1}{3}$. That is,

$$\epsilon(x)=\tfrac{1}{3},\quad \epsilon(y)=\tfrac{1}{3},\quad \epsilon(z)=\tfrac{1}{3}.$$

Therefore the expectations of x, $x+y$, $x+y+z$ are respectively

$$\tfrac{1}{3}=\tfrac{6}{18};\quad \tfrac{1}{3}+\tfrac{1}{6}=\tfrac{5}{18};\quad \tfrac{1}{3}+\tfrac{1}{6}+\tfrac{1}{3}=\tfrac{11}{18};$$

[Another solution of this Question is given in Vol. LXVIII., p. 99.]

13822. (ROBERT W. D. CHRISTIE.)—Prove the following curious theorem, which is true for primes only. (Is there any other, except WILSON'S ?) :— $(2s)^p\{(2s)^p+1\}\equiv 1 \pmod p$,
where $s=\sin 18°$ and p any odd prime whatever.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let x^2 be any positive integer (x may be a surd) not divisible by p.

Then $$\left(\frac{x-1}{2}\right)^p=\frac{x^p-1+p\mathrm{X}}{2^p}\equiv\frac{x^p-1}{2^p}\pmod p;$$

therefore $$\left(\frac{x-1}{2}\right)^p+1=\frac{x^p-1+p\mathrm{X}+2^p}{2^p}\equiv\frac{x^p+1}{2^p}\pmod p,$$

since $$2^p\equiv 2\pmod p;$$

therefore $$\left(\frac{x-1}{2}\right)^p\left\{\left(\frac{x-1}{2}\right)^p+1\right\}\equiv\frac{x^{2p}-1}{2^{2p}}\pmod p\equiv\frac{x^2-1}{2^{2p}}\pmod p,$$

since $$x^{2p}-x^2(x^2)^{p-1}\equiv x^2\pmod p.$$

Now let $x=\sqrt{5}$; therefore $\frac{x-1}{2}=2\sin 18°=2s$;

therefore $$(2s)^p\{(2s)^p+1\}\equiv\frac{4}{2^{2p}}\equiv\frac{1}{4^{p-1}}\pmod p.$$

Or $$4^{p-1}(2s)^p\{(2s)^p+1\}\equiv 1\pmod p;$$

therefore $(2s)^p\{(2s)^p+1\}\equiv 1 \pmod p$; since $4^{p-1}\equiv 1 \pmod p$,
except $p=2$, which proves the theorem.

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This theorem may be generalized as follows :—Let a, b, y be any positive integers, prime to p. Then

$$\left(\frac{x-y}{a}\right)^p = \frac{x^p-y^p+p\,.\,f(x,\,y)}{a^p} \equiv \frac{x^p-y^p}{a^p} \pmod p;$$

therefore $\left(\frac{x-y}{a}\right)^p + b^p = \frac{x^p-y^p+p\,.\,f(x,\,y)+a^p b^p}{a^p} \equiv \frac{x^p+y^p}{a^p} \pmod p$,

provided $(ab)^p \equiv 2y^p \pmod p$ or $ab \equiv 2y \pmod p$,

since $a^p \equiv a$, $b^p \equiv b$, $y^p \equiv y \pmod p$.

Then $\left(\frac{x-y}{a}\right)^p \left\{ \left(\frac{x-y}{a}\right)^p + b^p \right\} \equiv \frac{x^{2p}-y^{2p}}{a^{2p}} \pmod p \equiv \frac{x^2-y^2}{a^2} \pmod p$,

since $x^{2p} \equiv x^2$, $y^{2p} \equiv y^2$, $a^{2p} \equiv a^2 \pmod p$.

Fina $\left(\frac{x-y}{a}\right)^p \left\{ \left(\frac{x-y}{a}\right)^p + b^p \right\} \equiv 1 \pmod p$,

if $ab \equiv 2y$ and $x^2-y^2 \equiv a^2 \pmod p$.

[The Proposer remarks :—"The above theorem may also be proved by means of the 'continuant' series (*v.* Quest. 13480, 13533, &c.), which appears to be a series fertile in theorems concerning primes. Another theorem of the same kind is $2^p(t^p-s^p) = 1 \pmod p$, where p is any prime whatever, $t = \cos 36°$, $s = \sin 18°$.]

13712. (W. C. Stanham, B.A.)—If the centroid of a triangle is a fixed point, (*a*) its angular points lie on a fixed conic if its sides touch a fixed conic, (*b*) its sides touch a fixed conic if its angular points lie on a fixed conic.

Solution by the Proposer.

In the following (*b*) will be proved and (*a*) deduced. Take the fixed centroid as origin, the axis of x parallel to the transverse axis of the circumscribing conic, and the axis of y at right angles to it. Let the equation of the conic be

$$ax^2+by^2+2gx+2fy+c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The middle point (x', y') of the chord whose equation is $lx+my+1 = 0$ is

$$x' = (flm-gm^2-bl)/(bl^2+am^2),$$

with a similar expression for y'.

If the point $x = -2x'$, $y = -2y'$ is on the conic, we obtain, by means of (1), an equation of the fourth degree in l and m, but this will be found to contain the factor (bl^2+am^2); therefore the tangential equation of the envelope of the sides of triangles, having the origin as centroid, whose angular points lie on (1) is of the second degree, and the envelope is a conic. This proves (*b*).

Since, if a triangle is reciprocated with respect to its centroid, the reciprocal triangle has the same centroid, (*a*) follows.

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13642. (H. W. Curjel, M.A.)—Find the equation to the surface described by a point the sum of whose distances from three fixed points is constant.

Solution by Rev. J. Cullen, B.A.

Let A, B, C be the three given points, and the plane in which they lie $z = 0$. Then, if A is the origin, the coordinates of B and C are $(a_1, 0, 0)$ and $(a_2, b_2, 0)$, the line AB being the axis of x.

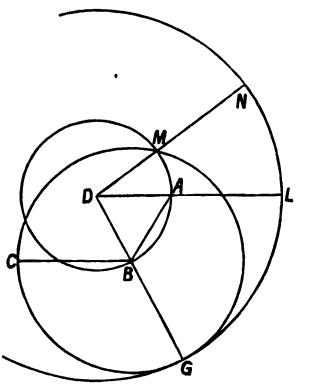
Hence we have $$r + (r^2 + \alpha^2)^{\frac{1}{2}} + (r^2 + \beta^2)^{\frac{1}{2}} = c,$$

where $r^2 = x^2 + y^2 + z^2$, $\alpha^2 = a_1(a_1 - 2x)$, $\beta^2 = a_2(a_2 - 2x) + b_2(b_2 - 2y)$.

Therefore $$64r^2(r^2 + \alpha^2)(r^2 + \beta^2) - \left[\left\{c^2 - (3r^2 + \alpha^2 + \beta^2)\right\}^2 - 4\left\{3r^4 + 2r^2(\alpha^2 + \beta^2) + \alpha^2\beta^2\right\}\right]^2 = 0,$$

a surface of the eighth degree.

13863. (The late R. Potts, LL.D.)—In the diagram (Potts' *Euclid*) of Euc. I. 2, with centre D and radius DA, describe a circle intersecting the circle CGH in the point M. Join DM, and produce it to meet the circumference of the circle GKL in N; then MN shall be equal to AL. What is the condition that this construction may be possible?



Solution by Rev. T. Mitcheson, B.A.

From a glance at the figure, it appears that the proposition will always be true as long as the circle MAB intersects the smaller of the other two circles, and this must be the case except when DB, *i.e.*, $AB < \frac{1}{2}BC$.

[The Rev. Dr. Frenth adds the following remarks:—"When AB is equal to the half of BC, the circles CGM and MAB touch at M in BD produced, and MN is still equal to AL or BC. Also the circles CGM and GLN touch at G in DB produced."]

13567. (R. W. D. Christie.)—Prove that an infinite number of integral values of a, b, A, B can be found to satisfy the equation

$$(3a^2 + 2ab + 2b^2)^2 \equiv (3A^2 + 2AB + 2B^2) + 2.$$

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Solution by the PROPOSER.

We have

$$(\omega_2+\omega_3)^{2n}+(\omega_4+\omega_5)^{2n} = 3.7.18.47.123, \text{ \&c., \&c.} = 3a^2+2ab+2b^2.$$

It will be found on trial that this series possesses the somewhat remarkable property $(a_n)^2 = a_{2n-1}+2$;

vide the solutions of Quests. 13391, 13533, 13480 for the connexion.

14002. (R. F. MUIRHEAD.)—Find an algebraic function $\phi(x)$ which has the following properties connected with the series of numbers

$$3, 4, 7, 12, 19, \ldots, n^2+3.$$

(1) If applied to any one of these numbers except the first, it gives rise to the preceding one, *e.g.*, $\phi(12) = 7$, $\phi(7) = 4$; and applied to 3 it gives $\phi(3) = 4$.

(2) If applied to any number lying between two successive terms of the series (except 3 and 4), it gives a number in the next interval below; but, if applied to a number a lying between 3 and 4, the result is another number b lying between 3 and 4, such that $\phi(a) = b$ and $\phi(b) = a$.

(3) If applied to any of the numbers $3\frac{1}{4}$, $5\frac{1}{4}$, $9\frac{1}{4}$, ..., $n(n+1)+3\frac{1}{4}$ ... except the first, it reproduces the next below; and applied to $3\frac{1}{4}$ it makes $\phi(3\frac{1}{4}) = 3\frac{1}{4}$.

Solution by H. W. CURJEL, M.A.; *and* Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $x = n^2+3$; then $n = \sqrt{(x-3)}$; therefore

$$\phi(x) = (n-1)^2+3 = x-2\sqrt{(x-3)}+1,$$

and ϕ is easily seen to satisfy the other conditions of the Question.

14007. (Rev. T. MITCHESON, B.A.)—If

$$\frac{-x+y+z}{x+y-z}+\frac{x-y+z}{-x+y+z}+\frac{x+y-z}{x-y+z} = \frac{x}{z}+\frac{y}{x}+\frac{z}{y},$$

then $x+y+z = 0$.

Solution by R. KNOWLES, B.A.; Rev. T. MITCHESON, B.A.; Rev. T. ROACH, M.A.; *and others.*

If $x+y+z = 0$, then

$$y+z = -x;\ x+y = -z;\ x+z = -y.$$

Substituting these values on the left-hand side of the equation, it reduces to

$$x/z+y/x+z/y = 0.$$

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14010. (THOMAS T. SMITH.)—A and B are two points on the same side of a given straight line XY such that the distance between A and B is less than the sum of their perpendicular distances from XY. Find the foci of two parabolas passing through A and B and having XY for their common directrix.

Solution by LIONEL E. REAY, B.A.; *the* PROPOSER; *and others.*

If a circle is drawn with A as centre to touch XY, then all the parabolas with foci on this circle, and with XY as their common directrix, pass through A. Similarly for the point B. Hence the foci of the two parabolas are the intersections of the two circles.

[This Question is virtually the converse of 13902 (Professor FRANZ ROGEL), Vol. LXX., p. 62.]

13964. (G. H. HARDY.)—Prove

$$\int_0^{\frac{1}{4}\pi} \tanh^{-1}(p\tan^2 x)\,dx = \tan^{-1}\sqrt{p}\,\tanh^{-1}\sqrt{p} - \tfrac{1}{4}\pi\tanh^{-1}p,$$

$$\int_0^{\lambda} \log(1+\tan\lambda\tan x)\,dx = \lambda\log\sec\lambda,$$

$$\int_0^{\frac{1}{4}\pi} (x\cot x)^2\,dx = \tfrac{3}{8}\int_0^1 \left(\frac{\sin^{-1}x}{x}\right)^2 dx = \pi\log 2 - \frac{\pi^2}{24}.$$

Solution by Professor UMES CHANDRA GHOSH, M.A.

(i.) Let $$u = \int_0^{\frac{1}{4}\pi} \tanh^{-1}(p\tan^2 x)\,dx,$$

therefore $$2\frac{du}{dp} = \int_0^{\frac{1}{4}\pi} \frac{\tan^2 x}{1+p\tan^2 x}\,dx + \int_0^{\frac{1}{4}\pi} \frac{\tan^2 x}{1-p\tan^2 x}\,dx$$

$$= \frac{1}{\sqrt{p}}\,\frac{1}{1-p}\tan^{-1}\sqrt{p} + \frac{1}{\sqrt{p}}\,\frac{1}{1+p}\,\tfrac{1}{2}\log\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right) - \frac{2}{1-p^2}\,\tfrac{1}{4}\pi.$$

Let $$\tan^{-1}\sqrt{p} = \theta;$$

then $$2u = \int \frac{2\theta}{\cos 2\theta}\,d\theta + \int \log\left(\frac{1+\tan\theta}{1-\tan\theta}\right) d\theta - \pi\int \frac{d(\tan\theta)}{1-\tan^2\theta}$$

$$= \theta\log\left(\frac{1+\tan\theta}{1-\tan\theta}\right) - \int \log\left(\frac{1+\tan\theta}{1-\tan\theta}\right) d\theta + \int \log\left(\frac{1+\tan\theta}{1-\tan\theta}\right) d\theta$$

$$-\tfrac{1}{4}\pi\log\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)$$

$$= 2\tanh^{-1}\sqrt{p}\,\tan^{-1}\sqrt{p} - \tfrac{1}{2}\pi\tanh^{-1}\sqrt{p};$$

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therefore $u = \tanh^{-1}\sqrt{p}\tan^{-1}\sqrt{p} - \frac{1}{4}\pi\tanh^{-1}p$,
the constant being zero.

(ii.) $$I = \int_0^\lambda \log(1+\tan\lambda\tan x)\,dx = \int_0^\lambda \log\{1+\tan\lambda\tan(\lambda-x)\}\,dx$$
$$= \int_0^\lambda \log\left(\frac{\sec^2\lambda}{1+\tan\lambda\tan x}\right)dx = \int_0^\lambda \log(\sec^2\lambda)\,dx - I;$$
therefore $2I = 2\lambda\log(\sec\lambda)$; therefore $I = \lambda\log(\sec\lambda)$.

$$\int_0^1 \left(\frac{\sin^{-1}x}{x}\right)^3 dx = \tfrac{1}{2}\int_0^1 \frac{3(\sin^{-1}x)^2}{x^2\sqrt{(1-x^2)}}\,dx \text{ [by integration by parts]}$$
$$= \tfrac{3}{2}\int_0^{\frac{1}{4}\pi} \theta^2\operatorname{cosec}^2\theta\,d\theta - \tfrac{1}{2}\cdot\tfrac{1}{8}\pi^3 \text{ [assuming } \sin\theta = x\text{]};$$
therefore $$\tfrac{3}{2}\int_0^1 \left(\frac{\sin^{-1}x}{x}\right)^3 dx = \int_0^{\frac{1}{4}\pi} \theta^2(\operatorname{cosec}^2\theta - 1)\,d\theta = \int_0^{\frac{1}{4}\pi} x^2\cot^2 x\,dx,$$
and $$\int_0^{\frac{1}{4}\pi} x^2\cot^2 x\,dx = \int_0^{\frac{1}{4}\pi} x^2(\operatorname{cosec}^2 x - 1)\,dx = \int_0^{\frac{1}{4}\pi} x^2\operatorname{cosec}^2 x\,dx - \frac{\pi^3}{24}.$$
But $$\int_0^{\frac{1}{4}\pi} x^2\operatorname{cosec}^2 x\,dx = \int_0^{\frac{1}{4}\pi} \cot x\,.\,2x\,dx \text{ [by integration by parts]}$$
$$= -2\int_0^{\frac{1}{4}\pi} \log(\sin x)\,dx = \pi\log 2;$$
therefore $$\int_0^{\frac{1}{4}\pi} x^2\cot^2 x\,dx = \pi\log 2 - \frac{\pi^3}{24}.$$

13976. (Rev. J. Cullen, B.A.)—In Quest. 13921, if R_A, &c., are the radii of T_A, &c., prove that

(1) $R_A\sin A = R_B\sin B = R_C\sin C = R'\sin\theta$;

(2) the join of the centre of the circle DEF [*i.e.*, $a\sec(A-\theta) = \ldots = \ldots$] and the radical centre $a\sin(A+\theta)/\sin 2A = \ldots = \ldots$ passes through the orthocentre of ABC.

Solution by Professor Sanjána, M.A.

As the triangles AFE′ and ABC are similar,
$$R_A : R = FE' : BC = 2R'\sin\theta : 2R\sin A;$$
hence $$R_A\sin A = R'\sin\theta;$$
so for R_B and R_C.

The trilinear coordinates of the centre of the circle DEF are $R'\cos(A-\theta)$ $R'\cos(B-\theta)$, $R'\cos(C-\theta)$; and those of the radical centre of T_A, T_B, T_C are $k\sin 2A/\sin(A+\theta)$, $k\sin 2B/\sin(B+\theta)$, $k\sin 2C/\sin(C+\theta)$.

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Hence the equation of their join is

$$\Sigma\, aR'k \left\{ \frac{\sin 2C \cos (B-\theta)}{\sin (C+\theta)} - \frac{\sin 2B \cos (C-\theta)}{\sin (B+\theta)} \right\} = 0,$$

i.e.,
$$\Sigma\, a \sin (A+\theta) \{\sin 2C \cos (B-\theta) \sin (B+\theta) - \sin 2B \cos (C-\theta) \sin (C+\theta)\} = 0,$$

i.e.,
$$\Sigma\, a \cos A \sin (A+\theta) \sin (B-C) = 0.$$

As the orthocentre is $a \cos A = \beta \cos B = \gamma \cos C$,

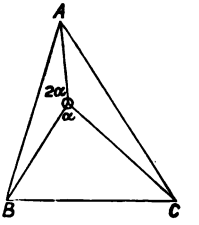
and
$$\Sigma \sin (A+\theta) \sin (B-C) = 0,$$

the join evidently goes through this point.

13871. (W. J. C. Miller, B.A.)—In order to determine the position of a rock, which was covered at high water, the angles subtended at it by the distances between three headlands were observed, and found to be 151° 2′ 42″, 133° 25′ 57″, 75° 31′ 21″; and it was known, from a previous survey, that the distances which subtended these angles were respectively 9870, 10857, 8883 yards. Find the distance of the rock from each of the headlands; and show the numerical calculation.

Solution by C. Joss, M.A.

Let A, B, C be the headlands and O the rock,



and
$$BOC = 75° 31′ 21″ = \alpha;$$

the distances are as 10 : 11 : 9. These being known, the angles are known as

$$ABC = 70° 31′ 44″;$$

therefore
$$BAO + BCO = 62° 54′ 13″,$$

$$\frac{\sin 2\alpha}{10 \times 987} = \frac{\sin BAO}{OB} \quad \text{and} \quad \frac{\sin \alpha}{9 \times 987} = \frac{\sin BCO}{OB}.$$

Dividing,
$$\tfrac{2}{3} \cos \alpha = \frac{\sin BAO}{\sin BCO} = \frac{\sin (62° 54′ 13″ - BCO)}{\sin BCO},$$

which gives
$$\cot BCO = \cot 44° 30′ 49″;$$

therefore
$$OBC = 59° 57′ 50″ \quad \text{and} \quad ABO = 10° 33′ 54″$$

hence we get the required distances as

$$\frac{\sin 44° 30′ 49}{BO} = \frac{\sin \alpha}{9 \times 987},$$

which gives
$$BO = 6431{\cdot}9.$$

Similarly,
$$AO = 3738{\cdot}0 \quad \text{and} \quad OC = 8127{\cdot}4.$$

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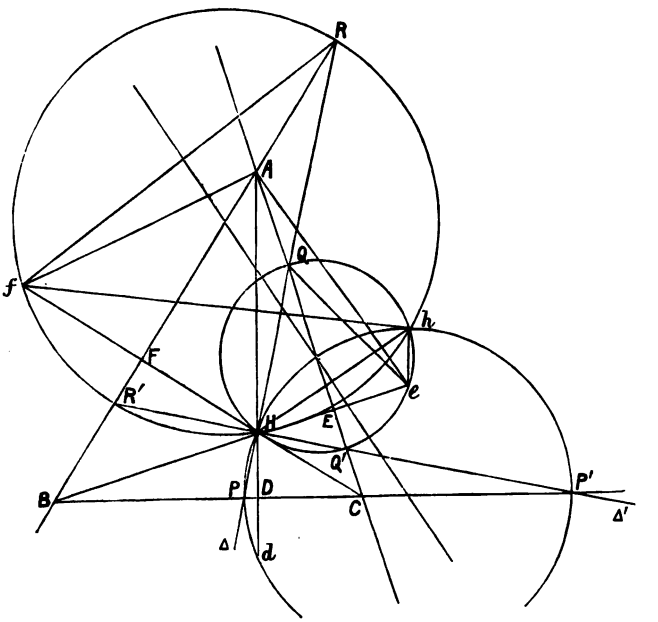
14111. (Professor A. Droz-Farny.)—Menons par l'orthocentre H d'un triangle ABC deux transversales Δ et Δ′ perpendiculaires l'une sur l'autre. Il s'agit de démontrer que ces perpendiculaires déterminent sur chaque côté du triangle un segment dont les points milieux α, β, γ sont en ligne droite.

Solutions (1) *by* C. E. Hillyer, M.A.; (2) *by* Professor Sanjána, M.A.

(1) Let the transversals Δ, Δ′ meet the sides in PQR, P′Q′R′ respectively. Let the circles RHR′, QHQ′ meet again in h, and let CH meet AB in F and the circle RHR′ in f, and BH meet CA in E and the circle QHQ′ in e.



Then, since $Ff = FH$ and $Ee = EH$, e and f are on the circumcircle.

Also $\angle fhH = fRH = 2FRH$ and $\angle ehH = eQH = 2EQH$;

therefore $\angle fhe = 2A$ and $\angle fAe = 2FAH + 2EAH = 2A$;

therefore $\angle fhe = fAe$; therefore h is on the circumcircle.

Similarly the circle PHP′ passes through the same point h on the circumcircle.

Therefore the circles are coaxal and their centres collinear.

Cor.—Since the mid-point of Hh is on the nine-points circle and the line of collinearity is the perpendicular to Hh through this point, it

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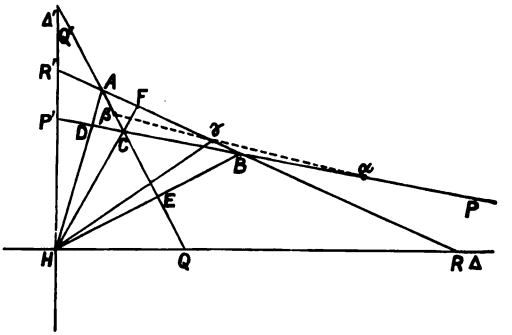
touches the ellipse of which H is a focus and the nine-points circle the auxiliary circle; *i.e.*, the inscribed ellipse with foci H and O.

(2) This proposition depends on the following important theorem, which I have not seen in print:—If three points A, B, C be referred to two axes meet-in O, the intersections of BC, CA, AB with the isogonal conjugates with regard to the axes of OA, OB, OC, respectively are collinear.



In the present case, if AB meet the axes in R and R', and the mid-point γ of RR' be joined to H, we have

$$\angle\gamma HR = R'RH = CHR';$$

thus Hγ and HC are isogonally conjugate. If BC, CA intercept PP', QQ' on the axes, and α, β be the mid-points of the intercepts, we have, similarly, Hα, Hβ conjugate to HA, HB. Hence the proposition.

[Professor A. Droz-Farny remarks: "Le théorème cité par Monsieur le Professeur Sanjána n'est qu'un cas particulier du théorème bien connu: Si d'un même point on mène des droites aux trois sommets d'un triangle et trois autres droites formant avec ces premières trois couples en involution, ces trois droites iront rencontrer les côtés opposés du triangle en trois points situés en ligne droite. (Voir Chasles, *Geométrie Supérieure*, p. 247.) Voici une troisième démonstration de la question que j'avais proposée: Le trois côtés du triangle ABC et les deux transversales orthogonales menées par son orthocentre sont tangentes à une même parabole et la question proposée devient un cas particulier de la proposition connue: Les points milieux des segments des tangentes à une parabole compris entre deux tangentes fixes sont sur une ligne droite qui est elle même tangente à la courbe."]

14104. (Professor Elliott, F.R.S.)—X, Y, Z, O_1, O_2 are five points on any cubic; O_1X, O_1Y, O_1Z meet the cubic again in A, B, C respectively, and O_2X, O_2Y, O_2Z in A', B', C'. Prove that the points P, Q, R, where BC' and B'C, CA' and C'A, AB' and A'B respectively intersect, also lie on the cubic; and that PX, QY, RZ meet in a point on the cubic. Also prove conversely that, if O be a tenth point on any cubic of the pencil which passes through six points A, B, C, A', B', C' and through P, Q, R derived from the six as above, and if PO, QO, RO meet the

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cubic again in X, Y, Z respectively, then AX, BY, CZ meet in a point on the cubic, as also do A′X, B′Y, C′Z.

Solution by the Rev. J. Cullen.

Let the lines O_1X, O_1Y, O_1Z be α_1, β_1, γ_1, and O_2X, O_2Y, O_2Z be α_2, β_2, γ_2. Then the equation of the cubic is

$$p\alpha_1\beta_2\gamma_2 + q\alpha_2\beta_1\gamma_1 + r\beta_1\alpha_2\gamma_2 + s\beta_2\alpha_1\gamma_1 + t\gamma_1\alpha_2\beta_2 + u\gamma_2\alpha_1\beta_1 = 0,$$

with $\alpha_1+\beta_1+\gamma_1 = 0$ and $\alpha_2+\beta_2+\gamma_2 = 0$.

Hence we have the intersections of the lines as follows:

$r\gamma_2+q\gamma_1-t\beta_2 = 0$, $\alpha_1 = 0$ ..... A; $p\gamma_2+s\gamma_1-u\beta_1 = 0$, $\alpha_2 = 0$ ..... A′;

$t\alpha_2+s\alpha_1-p\gamma_2 = 0$, $\beta_1 = 0$ ..... B; $r\alpha_2+u\alpha_1-q\gamma_1 = 0$, $\beta_2 = 0$ ..... B′;

$p\beta_2+u\beta_1-r\alpha_2 = 0$, $\gamma_1 = 0$ ..... C; $t\beta_2+q\beta_1-s\alpha_1 = 0$, $\gamma_2 = 0$ ..... C′.

Therefore
$$BC' \equiv t\alpha_2+s\alpha_1-p\gamma_2-q\beta_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1),$$
$$B'C \equiv r\alpha_2+u\alpha_1-p\beta_2-q\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

Now the cubic may be written in the form

$$BC'\beta_2\gamma_1 + B'C\beta_1\gamma_2 = 0.$$

Therefore P lies on the curve; in like manner, also do Q and R.

Adding (1) and (2), we have $(p+r+t)\,\alpha_2+(q+s+u)\,\alpha_1 = 0 \equiv PX$, since it passes through P and X. Similarly,

$$(p+r+t)\,\beta_2+(q+s+u)\,\beta_1 = 0 \equiv QY,$$
$$(p+r+t)\,\gamma_2+(q+s+u)\,\gamma_1 = 0 \equiv RZ;$$

adding, we see that these lines meet in a point, and it lies on the cubic; for, eliminating α_2, β_2, and γ_2, we find the result vanishes.

To prove the converse, let the lines BC′, B′C, A′C, AC′, AB′, and A′B be respectively π_1, π_2, ρ_1, ρ_2, σ_1, and σ_2. Then the cubic is of the form
$$\pi_2\rho_2\sigma_2 - k\pi_1\rho_1\sigma_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3).$$

Let O, any tenth point, be the intersection of the lines $\pi_2-\lambda\pi_1 = 0$, $\rho_2-\mu\rho_1 = 0$, $\sigma_2-\nu\sigma_1 = 0$; we have then $k = \lambda\mu\nu$. Now $OP \equiv \pi_2-\lambda\pi_1$ intersects (3) on the conic $\rho_2\sigma_2-\mu\nu\rho_1\sigma_1 = 0$. Hence X is the intersection of $\rho_2-\nu\sigma_1 = 0$ and $\sigma_2-\mu\rho_1 = 0$. Therefore we have as follows:

$\pi_2-\lambda\pi_1 = 0 \equiv OP$, $\rho_2-\mu\rho_1 = 0 \equiv OQ$, $\sigma_2-\nu\sigma_1 = 0 \equiv OR$,

$\rho_2-\nu\sigma_1 = 0 \equiv AX$, $\sigma_2-\lambda\pi_1 = 0 \equiv BY$, $\pi_2-\mu\rho_1 = 0 \equiv CZ$,

$\sigma_2-\mu\rho_1 = 0 \equiv A'X$, $\pi_2-\nu\sigma_1 = 0 \equiv B'Y$, $\rho_2-\lambda\pi_1 = 0 \equiv C'Z$,

Now, adding the equations of AX, BY, and CZ, we see that these lines meet in a point which is on the curve, since they satisfy (3). Similarly for A′X, B′Y, and C′Z.

14163. (I. Arnold.)—Given the length l of an inclined plane, and its height h. A body descends from the top of the plane. Find at what distance from the top of the plane the body will begin to *describe* a space equal to the height in the same time that it would have fallen perpendicularly through the height.

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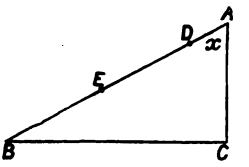
Solution by the PROPOSER ; *and* Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let l = length of plane AB, and h = AC the height; also let x = AD, the distance from top of plane to where the body begins to descend through DE = AC, and in the same time that it would fall perpendicularly through AC, and let g = gravity ; then



$$\left(\frac{2(x+h)}{\sin B \cdot g}\right)^{\frac{1}{2}}$$

= the time it descends through $h+x$ = AE;

and $\left(\frac{2x}{\sin B \cdot g}\right)^{\frac{1}{2}}$ = time it descends through x.

Consequently $\left(\frac{2(x+h)}{\sin B \cdot g}\right)^{\frac{1}{2}} - \left(\frac{2x}{\sin B \cdot g}\right)^{\frac{1}{2}}$ = time of descent through DE;

and Time through AC = $\left(\frac{2h}{g}\right)^{\frac{1}{2}}$;

or $$\left(\frac{2(x+h)}{\sin B \cdot g}\right)^{\frac{1}{2}} = \left(\frac{2x}{\sin B \cdot g}\right)^{\frac{1}{2}} + \left(\frac{2h}{g}\right)^{\frac{1}{2}}.$$

Whence, substituting for sin B its equal h/l, we get

$$hl^2 - 2h^2l + h^3 = 4xhl, \quad \text{or} \quad l^2 - 2hl + h^2 = 4xl;$$

therefore $$x = \text{AD} = (l-h)^2/(4l).$$

14156. (J. WATSON.)—In a given circle is a fixed chord equal to the radius, and also a point (not the centre). Describe a second circle that shall touch the given circle and the chord whilst passing through the given point.

Solution by A. E. LAYNG.

Let ABC be the given circle, CD the given chord, P the point.

From O, the centre of circle ABC, draw ON perpendicular to CD and produce to meet the circumference in A and B.

Join AP, cutting CD at K.

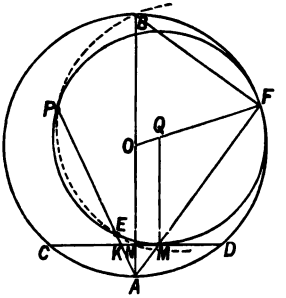
Through NPB describe a circle cutting AP at E.

From KD cut off KM such that

$$\text{KM}^2 = \text{KE} \cdot \text{KP}.$$

Through P, E describe a circle touching CD at M; this is the circle required.

For, let Q be its centre, join AM and produce to meet the circumference of ABC at F.



Then, since $\angle$ BFA is a right angle, the triangles BFA, MNA are

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similar; therefore $AM.AF = AN.AB.$

But, since B, P, E, N are points on a circle, therefore

$$AE.AP = AN.AB.$$

Hence $AM.AF = AE.AP$, and F is a point on the circle PEM. Join QF, OF, QM. Then, since QM is perpendicular to CD, therefore QM, ON are parallel: therefore $\angle QMF = OAF$.

But $\angle OAF = OFA$, and $\angle QMF = \angle QFM$.

Hence $\angle OFA = QFM$, and OQF is a straight line; therefore the circle PEM touches the circle ABC at F.

[The *length* of the chord CD need not have been given. Without this restriction, the question is a particular case of one of the "Tangencies," in which the given point is *within* the given circle.]

14080. (D. Biddle.)—Prove that N is factorizable $(N = 6n \pm 1)$ when $(N-1)^2 - 576\lambda = \square$ and $(N+1)^2 - 576\lambda = \square$ ..... (α, β), provided $\lambda > 0$ and integral, of the same value in each equation. [(α) represents the square of the difference of the factors, (β) represents the square of their sum. When N is prime, *both* equations hold good *only when* $\lambda = 0$.]

Solution by H. A. Webb.

Let $N \equiv xy$, x and y integral. x and y are both of the form $6n \pm 1$. Therefore

$$(x^2-1)(y^2-1) = (36l^2 \pm 12l)(36m^2 \pm 12m)$$
$$= 16 \times 3l(3l \pm 1) \times 3m(3m \pm 1) = M(16 \times 6 \times 6) = M(576).$$

Therefore $(x^2-1)(y^2-1) = 576\lambda,$

where λ is integral, and $\lambda > 0$ unless x or $y = 1$. Whence

$$(N-1)^2 - 576\lambda = (y-x)^2, \quad (N+1)^2 - 576\lambda = (x+y)^2 ..... (\alpha, \beta),$$

where λ is an integer.

Also, if $\lambda > 0$, neither x nor $y = 1$, and N is not prime. If N is prime, either x or $y = 1$, and $\lambda = 0$.

Note by the Proposer.

The theorem can be used for purposes of factorization as follows:—

$$\{\tfrac{1}{2}(N-1)\}^2 - 144\lambda = \{\tfrac{1}{2}(y-x)\}^2, \quad \{\tfrac{1}{2}(N+1)\}^2 - 144\lambda = \{\tfrac{1}{2}(y+x)\}^2.$$

It is thus clear that $\{\tfrac{1}{2}(N-1)\}^2 + 144$ has the same remainder as

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$\{\frac{1}{2}(y-x)\}^2 + 144$; and $\{\frac{1}{2}(N+1)\}^2 + 144$ the same remainder as $\{\frac{1}{2}(y+x)\}^2 + 144$.

By observation it is easily found that (in each case) such remainder is one of sixteen, which are as follows:—

0, 1, 4, 9, 16, 25, 36, 49, 52, 64, 73, 81, 97, 100, 112, 121.

With the exception of four, 52, 73, 97, 112, they are square numbers. Moreover, by observation again, it is found that the sequence of their occurrence is the same in each successive group of 72 squares,

$0, 1^2, 2^2, 3^2, \ldots, 72^2, 73^2, 74^2, 75^2, \ldots, 144^2, 145^2, 146^2, 147^2, \ldots$, &c.

And in each such group, after the commencing one whose remainder is 0, are 71, the remainders of which are arranged in exactly reversed order about another central 0, the first 35 leading up to this 0 in one way, the last 35 receding from it the other. Appended is a table giving for the squares of the first 72 numbers the remainders that belong to them, or, rather, against each remainder those numbers in the first group of 72 the squares of which belong to it (leaving that remainder on division by 144). The second and succeeding groups are formed by successive additions of 72 to the corresponding numbers in the first group. The required root of square is $72m$ + one of the roots opposite given remainder. Now, since augmenting the root of the larger of the two squares (of which N is the supposed difference) enables us to make the more rapid progress, we proceed to find the remainder of $\{\frac{1}{2}(N+1)\}^2 + 144$, by the following rule, which depends upon

$$\left(C + \frac{c}{12}\right)^2 = C^2 + \frac{cC}{6} + \frac{c^2}{144},$$

and upon

$$\frac{cC}{6} + \frac{c^2}{144} = C' + \frac{24c' + c^2}{144},$$

where C, C′ are quotients, and c, c' are remainders.

Rule I.—Take $\frac{1}{2}(N+1)$, and divide it by 12. Multiply the quotient by the remainder, and divide the product by 6. Multiply the remainder now found by 24, and add the square of the former remainder. If the total be greater than 144, subtract 144 from it. In either case, the remainder for use by Table I. results.

Next, observe that $\frac{1}{2}(y+x) > S$, where S = the integral portion of $\sqrt{N}$. Consequently, we begin with the root next above S that is indicated against the remainder in Table I., or that is found by addition of a multiple of 72 to one such root as explained further on.

The process, where N is large, may be greatly expedited by using, conjointly with Table I., the other tables which are appended, namely, Table II., which, giving the possible sums of the digits of the half sum of the factors for the several sums of the digits of N, enables us at once to check our calculation under Rule I.; Table III., which gives the possible final digits of the half-sum of the factors for the several final

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digits of N; and Table IV., which gives the possible bidigital endings of the same. Table III. suffices in many instances; but, as the use of Table IV., is similar, the following rules refer to the latter alone.

Rule II.—Having carried out Rule I., and obtained a remainder, observe the values placed against it in Table I., and see that the sums of their digits agree with those found by Table II. which affords a test.

Rule III.—Next, taking the bidigital ending of N, obtain from Table IV. the possible values of the bidigital ending of the half-sum of the factors.

Rule IV.—Now revert to Table I, and, observing that all the values against any remainder are all odd or all even, compare with them the correspondingly odd or even values found by Rule III., and secure the equality of the final digits of the former with those of the latter, by addition, where necessary, of the lowest required multiple of 72. But all the odd final digits, or all the even final digits (as the case may be) found by Rule III., must be obtained by addition of 72 repeatedly, if need be, to the same initial values, as given in Table I.

Rule V.—Next secure similar equality in the penultimate digits (odd or even) by successive addition of 360 (= 72 × 5).

Rule VI.—Raise the values thus obtained, by successive additions of 1800 (= 72 × 25), until each exceeds S. This leaves the sum of digits in each, and also the bidigital endings of each, unaffected.

Rule VII.—Now begin the trials by squaring in succession the values arrived at, then subtracting N to discover if a square results. Failing this, the leaps henceforth are by addition, to the several values, of 1800. (Before carrying out this rule, however, read the Note by Rev. J. CULLEN given on p. 99.)

N.B.—The tables of tridigital endings given in the *Messenger of Mathematics*, Vol. XXVIII. (as an Appendix to my paper on "Factorization," commencing p. 116), would enable us to proceed by leaps of 9,000, but from a somewhat greater number of possible values.

The following number set for factorization by Mr. H. J. WOODALL, A.R.C.S. (as Quest. 13800, *Educational Times*), will serve as a good example:—

N =	3258232531	
12)	1629116266	$= \frac{1}{2}(N+1)$
C =	135759688	$10 = c$
	10	
6)	1357596880	
	226266146	$4 = c'$
	24	
	96	
	100	$= c^2$
	196	
	144	
	Remainder = 52	

Sum of digits of N = 7.

Possible sums of digits of half sum of factors = 4, 5.

Bidigital ending of N = ... 31.

Possible bidigital endings (all even) of half sum of factors:

10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 34; 84; 16; 66.

S = 57080 = 31 . 1800 + 1280.

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Initial values by Table I.,

14, 22, 50, 58,

which all sum to 4 or 5, and are all even.

*14	*50	22	58
72	144	72	72
*86	*194	*94	*130
144	72	72	144
*230	*266	*166	*274
		144	72
		*310	*346

*50	*130	*230	*310	14	86
360	360	360	360	720	1080
*410	*490	*590	*670	*734	*1166
360	360	360	360		
*770	*850	*950	*1030	94	*166
360	360	360	360	1440	*266
*1130	*1210	*1310	*1390	*1534	
360	360	360	360	194	
*1490	*1570	*1670	*1750	1440	
				*1634	
				274	346
				360	720
				*634	*1066

We thus obtain the following 28 starting points:—

1310 + 55800	= 57110		410 + 57600	= 58010	
1390	,, = 57190	(1)	490	,, = 58090	(8)
1490	,, = 57290		590	,, = 58190	
1534	,, = 57334	(2)	634	,, = 58234	(9)
1570	,, = 57370	(3)	670	,, = 58270	(10)
1634	,, = 57434		734	,, = 58334	
1670	,, = 57470		770	,, = 58370	
1750	,, = 57550	(4)	850	,, = 58450	(11)
50 + 57600	= 57650		950	,, = 58550	
130	,, = 57730	(5)	1030	,, = 58630	(12)
166	,, = 57766	(6)	1066	,, = 58666	(13)
230	,, = 57830		1130		
266	,, = 57866		1166		
310	,, = 57910	(7)	1210		

On raising these by a multiple of 1800 above S, we start with 1310 + 55800, since S = 31 . 1800 + 1280, and we find that the 25th gives the correct half-sum of the factors, namely 58666, whence

N = 45121 . 72211.

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In the Question as it appeared in the *Educational Times*, it was further given that the factors were each of form $6n+1$. The half-sum must then have its digits sum to 4 (in the present instance). Those with 5 being crossed out, *the 13th trial is successful.*

Furthermore, as shown by Mr. CULLEN (see p. 99), the twelve trial-values preceding 58666, the successful one, can be eliminated without squaring, because they fail to comply with the following conditions, which can be applied one at a time to all values not eliminated previously:—

$N = 7m+1$;	therefore half-sum of factors	$= 7m \pm 1, 3$
$= 11m+4$;	,, ,, ,,	$= 11m \pm 2, 3, 4$
$= 13m+8$;	,, ,, ,,	$= 13m \pm 2, 3, 5$
$= 17m+2$;	,, ,, ,,	$= 17m \pm 1, 2, 3, 6, 7$
$= 19m+13$;	,, ,, ,,	$= 19m \pm 1, 6, 7, 9.$

N.B.—The difference between the two factors in this case is no less than 27090.

TABLE I.

Table of Remainders on dividing $\{\frac{1}{2}(N+1)\}^2$ or $\{\frac{1}{2}(N-1)\}^2$ by 144, and the initial values of the roots of the squares which they serve:—

0.	0, 12, 24, 36, 48, 60
1.	1, 17, 55, 71
4.	2, 34, 38, 70
9.	3, 21, 27, 45, 51, 69
16.	4, 32, 40, 68
25.	5, 13, 59, 67
36.	6, 18, 30, 42, 54, 66
49.	7, 25, 47, 65
52.	14, 22, 50, 58
64.	8, 28, 44, 64
73.	19, 35, 37, 53
81.	9, 15, 33, 39, 57, 63
97.	23, 31, 41, 49
100.	10, 26, 46, 62
112.	16, 20, 52, 56
121.	11, 29, 43, 61

The required root of square is $72m$ + one of the initial values opposite the given remainder.

TABLE II.

Table of sums of digits of N, and the possible sums of digits of the half-sum of its factors:—

1.	1, 8
2.	3, 6, 9
4.	2, 7
5.	3, 6, 9
7.	4, 5
8.	3, 6, 9

TABLE III.

Table of the final digits of N, and the possible final digits of the half-sum of its factors.

	odd	*even*
1.	1, 5, 9;	0, 4, 6
3.	3, 7;	2, 8
7.	1, 9;	4, 6
9.	3, 5, 7;	0, 2, 8

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Table IV.

Giving the Bidigital Endings of N (when not a multiple of 2, 3, or 5), and the Possible Bidigital Endings of the Half-sum of its two Factors.

01. 51; 01; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 99; 49.

03. 12, 32, 52, 72, 92; 02, 22, 42, 62, 82;
18, 38, 58, 78, 98; 08, 28, 48, 68, 88.

07. 14, 34, 54, 74, 94, 04, 24, 44, 64, 84;
16, 36, 56, 76, 96; 06, 26, 46, 66, 86.

09. 53; 03; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 97: 47.

11. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 94; 44; 56; 06.

13. 13, 33, 53, 73, 93; 03, 23, 43, 63, 83;
17, 37, 57, 77, 97; 07, 27, 47, 67, 87.

17. 11, 31, 51, 71, 91; 01, 21, 41, 61, 81;
19, 39, 59, 79, 99; 09, 29, 49, 69, 89.

19. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 12; 62; 38; 88.

21. 11; 61; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 39; 89.

23. 12, 32, 52, 72, 92; 02, 22, 42, 62, 82;
18, 38, 58, 78, 98; 08, 28, 48, 68, 88.

27. 14, 34, 54, 74, 94; 04, 24, 44, 64, 84;
16, 36, 56, 76, 96; 06, 26, 46, 66, 86.

29. 73; 23; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 77; 27.

31. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 34; 84; 16; 66.

33. 13, 33, 53, 73, 93; 03, 23, 43, 63, 83;
17, 37, 57, 77, 97; 07, 27, 47, 67, 87.

37. 11, 31, 51, 71, 91; 01, 21, 41, 61, 81;
19, 39, 59, 79, 99; 09, 29, 49, 69, 89.

39. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 92; 42; 58; 08.

41. 71; 21; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 79; 29.

43. 12, 32, 52, 72, 92; 02, 22, 42, 62, 82;
18, 38, 58, 78, 98; 08, 28, 48, 68, 88.

47. 14, 34, 54, 74, 94; 04, 24, 44, 64, 84;
16, 36, 56, 76, 96; 06, 26, 46, 66, 86.

49. 93; 43; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 57; 07.

51. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 74; 24; 76; 26.

53. 13, 33, 53, 73, 93; 03, 23, 43, 63, 83;
17, 37, 57, 77, 97; 07, 27, 47, 67, 87.

57. 11, 31, 51, 71, 91; 01, 21, 41, 61, 81;
19, 39, 59, 79, 99; 09, 29, 49, 69, 89.

59. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 72; 22; 78; 28.

61. 31; 81; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 19; 69.

63. 12, 32, 52, 72, 92; 02, 22, 42, 62, 82;
18, 38, 58, 78, 98; 08, 28, 48, 68, 88.

67. 14, 34, 54, 74, 94; 04, 24, 44, 64, 84;
16, 36, 56, 76, 96; 06, 26, 46, 66, 86.

69. 13; 63; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 37; 87.

71. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 14; 64; 36; 86.

73. 13, 33, 53, 73, 93; 03, 23, 43, 63, 83;
17, 37, 57, 77, 97; 07, 27, 47, 67, 87.

77. 11, 31, 51, 71, 91; 01, 21, 41, 61, 81;
19, 39, 59, 79, 99; 09, 29, 49, 69, 89.

79. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 52: 02; 98; 48.

81. 91; 41; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 59; 09.

83. 12, 32, 52, 72, 92; 02, 22, 42, 62, 82;
18, 38, 58, 78, 98; 08, 28, 48, 68, 88.

87. 14, 34, 54, 74, 94; 04, 24, 44, 64, 84;
16, 36, 56, 76, 96; 06, 26, 46, 66, 86.

89. 33; 83; 15, 35, 55, 75, 95; 05, 25, 45, 65, 85; 17; 67.

91. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 54; 04; 96; 46.

93. 13, 33, 53, 73, 93; 03, 23, 43, 63, 83;
17, 37, 57, 77, 97; 07, 27, 47, 67, 87.

97. 11, 31, 51, 71, 91; 01, 21, 41, 61, 81;
19, 39, 59, 79, 99; 09, 29, 49, 69, 89.

99. 10, 30, 50, 70, 90; 00, 20, 40, 60, 80; 32; 82; 18; 68.

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Further Note to the preceding Question (14080), *by* Rev. J. CULLEN.

The following method is intended to obviate the method of trials by squaring; *i.e.*, Rule VII., the principle involved being a generalization of that of casting out 9's as used in the Question.

Taking, then, $N = 9m+c = (9p+a)(9q+b)$, we see that the half-sums are of the form $(a+b)/2$, where $ab = c$, so, in general,

$$N = nm+c = (np+a)(nq+b) = \{n(p+1)-(n-a)\}\{n(q+1)-(n-b)\};$$

hence, besides $(a+b)/2$, we have $n-(a+b)/2$; so that $\pm(a+b)/2$ is a general half-sum. *Ex. gr.*, in Table II. 1, 8 is ± 1, with regard to 9, &c.

Taking $n = 7, 11, 13, 17, 19$ successively, the thirteen values for trial are reduced to the true one, viz., 58666.

By aid of the following table (which can easily be extended to include other moduli, if requisite) a very expeditious and exhaustive method can be employed, both for the determination of large primes and factorization. (See end of Vol., "Note on Mr. LAWRENCE's Method."

TABLE of possible Residues of the Half-sum of the Factors for each corresponding Residue of N, certain small primes being the Moduli. By Rev. J. CULLEN.

R	M (5)±	M (7)±	M (9)±	M (11)±	M (13)±	M (17)±	M (19)±	M (23)±
1.	0, 1	1, 3	1	1, 2, 4	0, 1, 2, 6	0, 1, 3, 4, 6	1, 5, 6, 8, 9	1, 2, 3, 5, 6, 7
2.	1	2, 3	0, 3	0, 4, 5	1, 4, 5	0, 1, 2, 6, 7	0, 3, 5, 7, 8	2, 5, 7, 8, 10, 11
3.	2	0, 2	—	1, 2, 5	0, 2, 4, 5	1, 2, 4, 6	0, 1, 2, 3, 8	1, 2, 3, 4, 7, 11
4.	0, 2	1, 2	2	2, 3, 4	0, 1, 2, 4	0, 2, 5, 6, 8	1, 2, 3, 7, 9	2, 4, 6, 9, 10, 11
5.	—	0, 3	3	3, 4, 5	1, 2, 3	1, 2, 3, 8	3, 4, 5, 7, 9	0, 3, 6, 8, 10, 11
6.	—	0, 1	—	0, 2, 3	3, 4, 6	2, 5, 6, 7	2, 5, 6, 7, 8	1, 3, 8, 9, 10, 11
7.	—	—	4	0, 1, 4	2, 4, 6	3, 4, 5, 7	2, 4, 7, 8, 9	0, 3, 4, 5, 6, 10
8.	—	—	0, 3	0, 1, 3	2, 3, 5	0, 2, 3, 4, 5	0, 3, 5, 6, 9	1, 3, 4, 7, 9, 10
9.	—	—	—	1, 3, 5	0, 3, 5, 6	0, 1, 3, 5, 8	1, 3, 4, 5, 8	2, 3, 5, 6, 8, 9
10.	—	—	—	0, 2, 5	0, 1, 3, 6	1, 3, 5, 6	0, 4, 6, 7, 8	0, 4, 6, 7, 8, 9
11.	—	—	—	—	1, 5, 6	3, 6, 7, 8	1, 3, 4, 6, 7	0, 1, 2, 6, 9, 11
12.	—	—	—	—	0, 3, 4, 5	2, 4, 5, 8	0, 2, 3, 4, 6	1, 4, 5, 6, 8, 9
13.	—	—	—	—	—	0, 2, 3, 7, 8	0, 1, 6, 7, 9	4, 5, 6, 7, 10, 11
14.	—	—	—	—	—	1, 4, 7, 8	0, 1, 2, 5, 7	0, 2, 3, 4, 7, 8
15.	—	—	—	—	—	0, 4, 6, 7, 8	0, 1, 4, 8, 9	0, 1, 2, 4, 8, 10
16.	—	—	—	—	—	0, 1, 4, 5, 7	1, 2, 4, 5, 6	1, 3, 4, 5, 8, 11
17.	—	—	—	—	—	—	2, 3, 6, 8, 9	0, 5, 7, 8, 9, 11
18.	—	—	—	—	—	—	0, 2, 4, 5, 9	1, 2, 6, 7, 8, 10
19.	—	—	—	—	—	—	—	0, 2, 3, 5, 9, 10
20.	—	—	—	—	—	—	—	0, 1, 3, 6, 7, 11
21.	—	—	—	—	—	—	—	0, 1, 2, 4, 5, 11
22.	—	—	—	—	—	—	—	0, 1, 5, 7, 9, 10

N.B.—It is remarkable that, though drawn up for the half-sums, they hold also for the half-differences by reading off the complementary residue (R) with respect to the modulus. *E.g.*, $\pm 2, 3, 5$ are the half-differences of 5M (13), being the half-sums of 8M (13). And this is important, for, if we know that one factor $X > \frac{\sqrt{2}+1}{\sqrt{2}-1} Y$, then the half-difference $> S$, and, using its trial-values, we get at the result quicker.

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14048. (Professor THOMAS SAVAGE.)—"No integral values of x, y, z can be found to satisfy the equation $x^n + y^n = z^n$, if n be an integer greater than 2." (FERMAT: W. W. ROUSE BALL'S *Short History of Mathematics*.) Prove the truth of this when x is any prime, y any positive integer not greater than x—the proof to hold for $n = 2$.

Solution by H. W. CURJEL, M.A.

If $x^n + y^n = z^n$, then $x^n = (z-y)(z^{n-1} + z^{n-2}y + \ldots + y^{n-1})$.

If $y \not> x$ and x is a prime, y cannot be equal to x, for then $2x^n$ would $= z^n$, which is clearly impossible if $n > 1$; therefore $y < x$. Also z is clearly less than $x + y$, since $(x+y)^n > x^n + y^n$; therefore $z - y < x$, and it is a factor of x^n; therefore $z - y = 1$; therefore $z = y + 1$; but $z > x$; therefore $y = z - 1 >$ or $= x$; therefore $y = x$, which is impossible.

14077. (Professor SANJÁNA.)—If the arcs joining the mid-points of the sides of a spherical triangle be each 90°, the sum of the squares of the cosines of the half-sides of the triangle shall be unity. (Suggested by Quest. 13661.)

Solution by R. TUCKER, M.A.; *and* Prof. A. DROZ-FARNY.

$$0 = \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \cos A,$$

and
$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Hence
$$2\cos^2 \tfrac{1}{2}a + 4\cos^2 \tfrac{1}{2}b \cos^2 \tfrac{1}{2}c = 1 + \cos b \cos c$$
$$= 2 + 4\cos^2 \tfrac{1}{2}b \cos^2 \tfrac{1}{2}c - 2(\cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c);$$

i.e.,
$$\Sigma \cos^2 \tfrac{1}{2}a = 1.$$

14124. (G. H. HARDY.)—Prove that

$$\int_0^{\frac{1}{4}\pi} \log \tan \phi \, d\phi = \tfrac{1}{2} \int_0^{\frac{1}{2}\pi} \log \tan \phi \, d\phi = -4 \int_0^{\frac{1}{4}\pi} \tanh^{-1} \tan \phi \, d\phi.$$

Solution by the PROPOSER.

We have
$$\int_0^u \frac{du}{\cos^2 u - p} = \frac{1}{\sqrt{\{p(1-p)\}}} \sinh^{-1} \frac{\sqrt{p} \sin u}{\sqrt{(\cos^2 u - p)}};$$

i.e., putting
$$p = \cos^2 x,$$

$$\int_0^u \frac{2 \cos x \sin x}{\cos^2 u - \cos^2 x} du = 2 \tanh^{-1} (\tan u \cot x),$$

whence
$$\int_{x_0}^{x'_0} \int_0^{u_0} \frac{2 \cos x \sin x}{\cos^2 u - \cos^2 x} du\, dx = 2 \int_{x_0}^{x'_0} \tanh^{-1} (\tan u_0 \cot x) \, dx.$$

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Let $u_0 = \frac{1}{4}\pi$; then

$$\int_0^{\frac{1}{4}\pi} \log \frac{\cos^2 u - \cos^2 x'_0}{\cos^2 u - \cos^2 x_0}\, du = \int_{x_0}^{x'_0} \log \frac{1+\cot x}{1-\cot x}\, dx.$$

Put $x'_0 = \frac{1}{3}\pi$, $x_0 = \frac{1}{3}\pi$, and $\cos^2 u - \frac{1}{4} = \dfrac{\sin 3u}{4 \sin u}$,

and $$\int_0^{\frac{1}{4}\pi} \log \tan x\, dx = -\sum_0^{\infty} \frac{1}{(2n+1)^2} = -\Delta, \text{ say,}$$

when we get, after some reduction,

$$\int_0^{\frac{1}{3}\pi} \log \tan x\, dx = -\tfrac{3}{8}\Delta,$$

and, writing $\frac{1}{4}\pi - y = x$,

$$-\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \log \frac{1-\tan y}{1+\tan y}\, dy = -\tfrac{3}{8}\Delta,$$

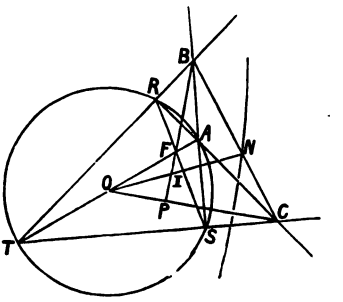
whence $$\int_0^{\frac{1}{2}\pi} \log \frac{1+\tan y}{1-\tan y}\, dy = 2\int_0^{\frac{1}{2}\pi} \tanh^{-1} \tan y\, dy = \tfrac{1}{8}\Delta,$$

which proves the result.

14161. (W. C. Stanham, B.A.)—If the tangents to a conic from any point on one of its directrices intersect any fixed straight line at points P and Q, the other two tangents to the conic from P and Q intersect on a fixed straight line.

Solution by the Proposer.

The theorem follows by reciprocation from the following theorem:—The straight lines—CA, CT in the figure—joining a fixed point to the extremities of any diameter of a fixed circle meet the circle again at points the join of which passes through a fixed point.



For BC is the polar of F; therefore the pole of RS lies on BC. But, if I is the mid-point of RS, the pole of RS lies on OI; and O, I, N, being the mid-points of the diagonals of ARTS, are collinear. Therefore N, the locus of which is a fixed straight line parallel to BFP, the polar of C, is the pole of RS. Therefore RS passes through a fixed point.

13635. (The late Professor Wolstenholme, M.A., D.Sc.)—On the sides of a given triangle ABC are erected equilateral triangles aBC, a'BC, &c.; A, a being on the same side of BC, A, a' on opposite sides of BC,

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and similarly for b, b', c, c'. It is well known that $Aa = Bb = Cc$, $Aa' = Bb' = Cc'$; but prove the following properties:—(1) Aa, Bb, Cc are each equal to three times the difference of the semiaxes of the maximum ellipse inscribed in the triangle ABC. (2) Aa', Bb', Cc' are each equal to three times the sum of the semiaxes of the maximum inscribed ellipse; and to the altitude of the maximum equilateral triangle circumscribed about the given triangle. (3) The straight lines bisecting the angles between the three pairs Aa, Aa', Bb, Bb', Cc, Cc' are all parallel to the axes of the maximum inscribed ellipse. In fact, if G be the centre of this ellipse, A', B', C' the points of contact (the mid-points of the sides), and if on the straight line drawn through A' at right angles to BC be taken points α, α', such that $\alpha\alpha'$ is bisected in A', and is equal to the diameter parallel to BC, $G\alpha$ is parallel to $A\alpha$ and $= \frac{1}{3}A\alpha$; $G\alpha'$ is parallel to $A\alpha'$ and $= \frac{1}{3}A\alpha'$; and, by the well-known construction for the axes of an ellipse from a given pair of conjugate diameters, $G\alpha$ is equal to the difference, and $G\alpha'$ to the sum of the semiaxes, and $G\alpha$, $G\alpha'$ are equally inclined to the axes.

Solution by LIONEL E. REAY, B.A.

The maximum ellipse in the triangle touches the sides at the middle points, and its centre G is the centroid. Let its principal semiaxes be λ, μ. By the construction quoted in Question [for proof see SALMON'S *Conic Sections*, § 181 (*a*)], the principal axes bisect the angles between $G\alpha$, $G\alpha'$, and $G\alpha = \lambda - \mu$, $G\alpha' = \lambda + \mu$.

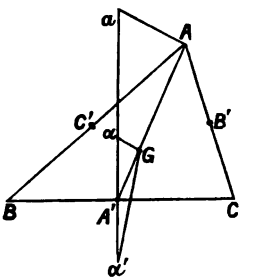

Fig. 1.

Diameter conjugate to $GA' = A'\alpha$ and is parallel to BC; then

$$\tfrac{1}{4}(B'C'/A'\alpha)^2 + \tfrac{1}{4} = 1;$$

therefore $B'C' = A'\alpha\sqrt{3}$;

therefore $BC = 2A'\alpha\sqrt{3}$;

therefore $\alpha A' = 3A'\alpha$;

therefore $A\alpha$ is parallel to $G\alpha$ and $= 3G\alpha = 3(\lambda - \mu)$.

Similarly, $A\alpha'$ is parallel to $9\alpha'$ and $= 3(\lambda + \mu)$; therefore the lines bisecting angles $\alpha A\alpha'$, &c., are parallel to the axes of the ellipse.

Again, if (Fig. 2)

$$BOC = COA = BOA,$$

then the lines through A, B, C, perpendicular to OA, OB, OC, are sides of maximum equilateral triangle about ABC.

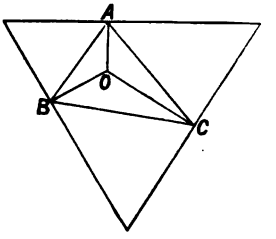

Fig. 2.

Also its altitude $= OA + OB + OC$.

Also

$$\begin{aligned} A\alpha'^2 &= a^2 + b^2 - 2ab\cos(C + 60) \\ &= a^2 + b^2 - ab\cos C + ab\sqrt{3}\sin C \\ &= \tfrac{1}{2}(\Sigma a^2 + 4\sqrt{3}\Delta), \end{aligned}$$

and $a^2 = OB^2 + OC^2 + OB \cdot OC$,

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therefore $\Sigma a^2 = 2\Sigma OA^2 + \Sigma OB \cdot OC$ and $2\Delta = (\Sigma OB \cdot OC)\frac{1}{2}\sqrt{3}$;
therefore $A a'^2 = (OA + OB + OC)^2 =$ square of altitude.

An alternative proof of part of the question is as follows:—
The equation to the ellipse is

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0,$$

λ, μ, are given by

$$\lambda\mu = (2\Delta H)/-K^{\frac{3}{2}} \quad \lambda^2 + \mu^2 = (-IH)/K^2,$$

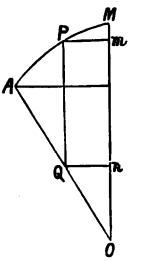
where $I = \Sigma a^2 + 2\Sigma bc \cos A = 2\Sigma a^2$.

$$H = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4 \qquad K = \begin{vmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -12;$$

therefore $\lambda^2 + \mu^2 = \frac{1}{18}(\Sigma a^2)$, $\lambda\mu = \frac{1}{9}(\Delta\sqrt{3})$,
therefore $(\lambda + \mu)^2 = \frac{1}{18}(\Sigma a^2 + 4\Delta\sqrt{3}) = \frac{1}{2}A a'^2$,
$(\lambda - \mu)^2 = \frac{1}{18}(\Sigma a^2 - 4\Delta\sqrt{3}) = \frac{1}{9}A a^2$.

14132. (I. Arnold.)—Describe a square in a given sector having two angular points on the arc and the other two on the radii.

Solution by Lt.-Col. Allan Cunningham, R.E.; *the* Proposer; *and* Rev. T. Mitcheson, B.A.



AM is half the arc; $AOM = \frac{1}{2}\alpha$ (half the angle of sector); PQ (a side of the square) $= x$; $Pm = Qn = \frac{1}{2}x$ (half sides of the square). Then

$$On = Qn \cot \tfrac{1}{2}\alpha,$$
$$Om = On + nm = \tfrac{1}{2}x \cot \tfrac{1}{2}\alpha + x,$$

and $Pm^2 + Om^2 = OP^2 = r^2$;
therefore $(\frac{1}{2}x)^2 + (\frac{1}{2}x \cot \frac{1}{2}\alpha + x)^2 = r^2$;
therefore $x^2 = r^2 + (\frac{5}{4} + \cot \frac{1}{2}\alpha + \frac{1}{4}\cot^2 \frac{1}{2}\alpha)$.

14069. (Rev. T. Roach, M.A.)—Sum to infinity the series

$$\tfrac{1}{2} - \tfrac{2}{5} + \tfrac{3}{8} - \tfrac{4}{11} + \tfrac{5}{14} - \ldots.$$

Solutions (1) *by the* Proposer; (2) *by* H. W. Curjel, M.A.

(1) $\tfrac{1}{2} - \tfrac{2}{5} + \tfrac{3}{8} - \tfrac{4}{11} + \tfrac{5}{14} - \ldots = \dfrac{1}{2.5} + \dfrac{1}{8.11} + \ldots = \frac{1}{3}(\tfrac{1}{2} - \tfrac{1}{5} + \ldots)$

to $2n$ terms.

Now $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$.

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Therefore $\frac{1}{3}x^2+\frac{1}{3}x^8-\ldots = -\frac{1}{6}\Sigma\omega^4\log(1+\omega x)$,
where $\omega = \cos\frac{1}{3}r\pi \pm i\sin\frac{1}{3}r\pi$, $r = 0\ldots3$;
$\frac{1}{5}x^5+\frac{1}{11}x^{11}+\ldots = \frac{1}{6}\Sigma\omega\log(1+\omega x)$.

Therefore $S = -\frac{1}{3}\cdot\frac{1}{6}\Sigma\omega(\omega^3-1)\log(1+\omega x)$.

Now $\omega^3-1=0$ or -2 as r is even or odd, and $1+\omega x=0$ if $r=3$ when $x=1$. Therefore $r=1$. Therefore

$$S = -\tfrac{1}{3}\cdot\tfrac{1}{6}(-2)\Sigma(\cos\tfrac{1}{3}\pi\pm i\sin\tfrac{1}{3}\pi)\log\left\{1+(\cos\tfrac{1}{3}\pi\pm i\sin\tfrac{1}{3}\pi)x\right\}$$
$$= \tfrac{1}{3}\left[\tfrac{1}{2}(1\pm i\sqrt{3})\log\left\{1+\tfrac{1}{2}(1\pm i\sqrt{3})x\right\}\right]$$
$$= \tfrac{1}{3}\left\{\tfrac{1}{2}\log(1+x+x^2)+\tfrac{1}{2}(i\sqrt{3})\log\frac{1+\frac{1}{2}(1+i\sqrt{3})x}{1+\frac{1}{2}(1-i\sqrt{3})x}\right\}$$
$$= \tfrac{1}{3}\left\{\tfrac{1}{2}\log 3+\tfrac{1}{2}(i\sqrt{3})\,2i\tan^{-1}\frac{\frac{1}{2}(\sqrt{3}x)}{1+\frac{1}{2}x}\right\} = \tfrac{1}{12}\log 3-\tfrac{1}{3}\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}$$
$$= \tfrac{1}{12}\log 3-\tfrac{1}{3}\sqrt{3}\cdot\tfrac{1}{3}\pi = \tfrac{1}{12}(\log 3-\pi/\sqrt{3}) \text{ to } 2n \text{ terms.}$$

Sum to $2n+1$ terms when n is infinite $= \frac{1}{3}+\frac{1}{12}\log(3-\pi/\sqrt{3})$.

(2) This is an oscillating series, the limiting values of the $(2n-1)$th and $2n$th terms being $\frac{1}{3}$ and $-\frac{1}{3}$ when n is ∞.

Sum to $2n$ terms when n is ∞

$$= \sum_{m=1}^{m=\infty}\left(\frac{2m-1}{6m-4}-\frac{2m}{6m-1}\right) = \sum_{m=1}^{m=\infty}\frac{1}{(6m-4)(6m-1)} = \tfrac{1}{3}\sum_{m=1}^{m=\infty}\left(\frac{1}{6m-4}-\frac{1}{6m-1}\right)$$
$$= -\tfrac{1}{3}\left\{\log 2+\omega\log(1+\omega)+\omega^2\log(1+\omega^2)\right\} \qquad (\text{where } \omega^3=1)$$
$$= \tfrac{1}{3}(\pi/\sqrt{3}-\log 2).$$

Sum to $2n+1$ terms when n is infinite
$= \frac{1}{3}+\frac{1}{3}(\pi/\sqrt{3}-\log 2)$.

14056. (G. W. Preston, B.A.)—If

$$\frac{x^4}{a}+\frac{y^4}{b} = \frac{1}{a+b} \quad \text{and} \quad x^2+y^2=1,$$

show that
$$\frac{x^{2n+2}}{a^n}+\frac{y^{2n+2}}{b^n} = \frac{1}{(a+b)^n}.$$

Solution by Professor T. Savage; R. Tucker, M.A.; *and others.*

Eliminating x^2 from the given equations and solving for y^2, we obtain

$$y^2 = \frac{b}{a+b}; \quad \text{therefore} \quad x^2 = \frac{a}{a+b}.$$

Hence
$$\left(\frac{x^2}{a}\right)^n x^2+\left(\frac{y^2}{b}\right)^n y^2 = \frac{x^2}{(a+b)^n}+\frac{y^2}{(a+b)^n} = \frac{1}{(a+b)^n}.$$

[See Wolstenholme's *Problems*, No. 159, Second Edition, 1878.]

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14159. (Rev, J. CULLEN.)—Prove (1) that the PASCAL line of the hexagon DD′EE′FF′, formed by the intersections of the sides of ABC with any "T" circle, is the radical axis of "T" and the circumcircle; (2) the intersections of the homologous sides of DEF and D′E′F′ are the vertices of a triangle homothetic with ABC, and the homothetic centre lies on KIEPERT's hyperbola.

Solution by G. N. BATES, B.A.

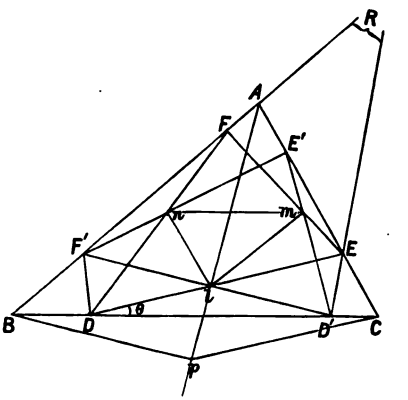
Let D′E meet AB in R.

Then, from similar triangles RD′B, REA,

$$RE.RD' = RA.RB;$$

therefore R is on radical axis of T_θ and circumcircle, which therefore coincides with PASCAL line of

DD′EE′FF′.



If DE, F′D′ meet in *l*, &c., the quadrilateral *nm*E′F is cyclic; therefore *nm* is parallel to BC; *i.e.*, *lmn* is homothetic with ABC. Now, if B*p* be drawn parallel to D′F′ to meet A*l* in *p*, then C*p* is parallel to DE, and *p* is vertex of isosceles triangle described on BC, with base angle θ; similarly for *q* and *r*; therefore A*p*, B*q*, C*r*, *i.e.*, A*l*, B*m*, C*n*, intersect in a point which lies on KIEPERT's hyperbola.

14072. (Professor NEUBERG).—Résoudre le système

$$\sin(y+z+u) = a\sin x, \qquad \sin(z+u+x) = b\sin y,$$
$$\sin(u+x+y) = c\sin z, \qquad \sin(x+y+z) = d\sin u.$$

Solution by H. A. WEBB.

Let $$x+y+z+u = \kappa.$$

Then $$\sin(\kappa - x) = a\sin x;$$

therefore $\tan x = \sin\kappa/(a+\cos\kappa)$, and three similar equations ..... (1).

Now
$$\tan\kappa = \frac{\Sigma\tan x - \Sigma\tan y\tan z\tan u}{1-\Sigma\tan x\tan y+\tan x\tan y\tan z\tan u}$$
$$= \frac{\Sigma\sin\kappa(a+\cos\kappa)(b+\cos\kappa)(c+\cos\kappa) - \Sigma\sin^3\kappa(d+\cos\kappa)}{(a+\cos\kappa)(b+\cos\kappa)(c+\cos\kappa)(d+\cos\kappa) - \Sigma\sin^2\kappa(a+\cos\kappa)(b+\cos\kappa)+\sin^4\kappa};$$

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therefore either $\sin\kappa = 0$, giving x, y, z, and u each a multiple of π... (2), or we have an equation in $\cos\kappa$ which reduces to

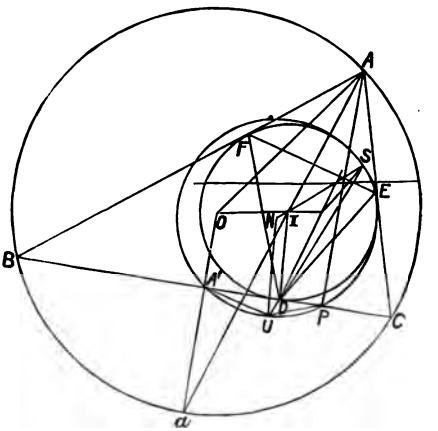
$$abcd - \Sigma ab + 1 - 2\cos\kappa \cdot \Sigma a - 4\cos^2\kappa = 0;$$

therefore $\cos\kappa = \frac{1}{4}\{-(a+b+c+d) \pm [a^2+b^2+c^2+d^2 - 2(ab+bc+ca+ad+bd+cd) + 4abcd + 4]^{\frac{1}{2}}\}$ ..... (3).

(2) is one set of solutions; the others are found by substituting the values of $\sin\kappa$, $\cos\kappa$ found from (3) in equations (1).

14075. (Professor A. Droz-Farny.)—Le cercle inscrit du triangle ABC touche les côtés en D, E, F. Soit S le point de contact de ce cercle avec le cercle d'Euler. Démontrez que la droite de Simson de S par rapport au triangle DEF est parallèle à la droite d'Euler de ce triangle. Théorème analogue pour les cercles ex-inscrits ?

Solution by C. E. Hillyer, M.A.



If O, N, I are the circum-, nine-points-, and in-centres, respectively, of ABC, and AI meets the circumcircle in a, A′ is the mid-point of BC so that OA′a is a straight line perpendicular to BC, AP is perpendicular to BC, and NU the parallel radius of the nine-points-circle, so that UDS is a straight line, then

$$\angle UA'P = UPA' = \tfrac{1}{2}(B \sim C) = OAa = OaA;$$

therefore the isosceles triangles UA′P, OAa are similar.

Also, by parallels, $A'D : DP = aI : IA$;

therefore the triangles UDA′, OIa are similar. Therefore

$$\angle OIa = UDA' = SDC = \tfrac{1}{2}SID,$$

since DC touches the circle DEF at D. Now it is known that the angle between the Simson-lines of two points is half the angle subtended by the two points at the centre; but the Simson-line of D with respect to the triangle DEF is the perpendicular from D on EF, and is therefore parallel to AI. Thus the angle between the Simson-line of S and AI

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$= \frac{1}{2}SID = OIa$. Therefore the SIMSON-line of S is parallel to OI, and it is known that OI passes through the orthocentre of DEF.

Note.—The fact that $\angle UDA' = OIa$ affords a very simple proof of FEUERBACH'S theorem, for it is easy to deduce from it that the point in which UD meets NI is on the circumference of the incircle.

14125, 14127, & 14129. (14125.) (DISCIPULUS.)—If $(1+2x)^m/(1-x)$ be expanded in ascending powers of x, prove that the coefficient of any power of x greater than $m-1$ is 3^m, m being a positive integer.

(14127.) (Rev. T. MITCHESON, B.A.)—Show that

$$(p+q)\,!/p\,!\,q\,! = 1 + pq + p(p-1)\,q(q-1)/(2\,!)^2 + p(p-1)(p-2)\,q(q-1)(q-2)/(3\,!)^2 + \ldots.$$

(14129.) (R. KNOWLES, B.A.)—Prove that the $(r+1)^{\text{th}}$ terms in the expansions of $(1+x)^{m/n}$ and $(1-x)^{-m/n}$ are respectively

$$\frac{(m+n-nr)\,(m+2n-nr)\,(m+3n-nr)\ldots \text{ to } r \text{ terms}}{n^r \,.\, r\,!};$$

$$\frac{(n-m-nr)\,(2n-m-nr)\,(3n-m-nr)\ldots \text{ to } r \text{ terms}}{n^r \,.\, r\,!}.$$

Solutions by Rev. T. WIGGINS, B.A., *and many others.*

(14125.)

$$(1+2x)^m/(1-x) = (1+2x)^m\,(1+x+x^2+\ldots)$$
$$= \left\{c_0 + c_1(2x) + c_2(2x)^2 + \ldots + c_m(2x)^m\right\}(1+x+x^2+\ldots).$$

In this product the coefficient of any term x^{m-1+k} is

$$(c_0 + c_1 2 + c_2 2^2 + c_3 2^3 + \ldots + c_m 2^m) = (1+2)^m = 3^m.$$

(14127.)
$$(1+x)^p = {}_pc_0 + {}_pc_1 x + {}_pc_2 x^2 + \ldots + {}_pc_p x^p;$$
$$(1+x)^q = {}_qc_0 + {}_qc_1 x + {}_qc_2 x^2 + \ldots + {}_qc_q x^q.$$

The coefficient of x^p in the product of these expansions = coefficient of x^p in the expansion of $(1+x)^{p+q}$. Hence

$$(p+q)\,!/p\,!\,q\,! = {}_pc_p \cdot {}_qc_0 + {}_pc_{p-1} \cdot {}_qc_{q-1} + {}_pc_{p-2} \cdot {}_qc_{q-2} + {}_pc_{p-3} \cdot {}_qc_{q-3} + \cdots$$
$$= 1 + pq + \frac{p.p-1.q.q-1}{(2\,!)^2} + \frac{p.p+1.p-2.q.q-1.q-2}{(3\,!)^2} + \ldots.$$

[Mr. G. D. MUGGERIDGE remarks: "By a theorem of GAUSS, if $F_1(\alpha, \beta, \gamma)$ be the value of the hypergeometric series when x is put equal to unity, then $F_1(\alpha, \beta, \gamma) = \frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)}$.

Put $\alpha = -p$, $\beta = -q$, $\gamma = 1$; this becomes

$$1 + (p\,.\,q)/(1\,!)^2 + \left\{p(p-1)\,q(q-1)\right\}/(2\,!)^2 + \ldots = (p+q)\,!/(p\,!\,q\,!).\text{"}]$$

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(14129.) (α) The $(r+1)^{\text{th}}$ term in $(1+x)^{m/n}$ is

$$\{[(m/n)(m/n-1)(m/n-2)\ldots(m/n-r+1)]/r!\}\,x^r$$
$$= \{[(m+n-r)(m+2n-r)(m+3n-r)\ldots \text{to } r \text{ terms}]/n^r r!\}\,x^r.$$

(β) The $(r+1)^{\text{th}}$ term in $(1-x)^{-m/n}$ is

$$\{[(-m/n)(-m/n-1)\ldots(-m/n-r+1)]/r!\}\,x^r$$
$$= \{[(n-m-nr)(2n-m-nr)(3n-m-nr)\ldots \text{to } r \text{ terms}]/n^r r!\}\,x^r.$$

[The Proposer of 14129 remarks:—"In each formula in this Question any term in the numerator can be obtained by adding an integer n to the preceding term, which will also be an integer."]

14070. (Francis Rountree, M.A.)—Solve the following sets of simultaneous equations:—

$$\left.\begin{aligned} x^2-5xy+5y^2+3x-5y+3 &= 0 \\ 3(x^2-2xy+4y^2)+2(x-2y)-1 &= 0 \end{aligned}\right\} \quad \begin{aligned} &\ldots\ldots\ldots\ldots\ldots (1), \\ &\ldots\ldots\ldots\ldots\ldots (2), \end{aligned}$$

and

$$\left.\begin{aligned} (1)\quad 2x^2+3xy &= 22x-18 \\ (2)\quad 2y^2-\ xy &= 22y-54 \end{aligned}\right\}.$$

Solution by H. J. Woodall, A.R.C.S.

From the first set

$$x^2+x(-5y+3)+(5y^2-5y+3)=0,\quad 3x^2+x(-6y+2)+(12y^2-4y-1)=0,$$

giving the determinant

$$\begin{vmatrix} +1, & -5y+3, & 5y^2-5y+3, & 0 \\ 0, & +1, & -5y+3, & +5y^2-5y+3 \\ +3, & -6y+2, & 12y^2-4y-1, & 0 \\ 0, & +3, & -6y+2, & 12y^2-4y-1 \end{vmatrix} = 0,$$

which reduces to $279y^4-420y^3+104y^2+8y+37=0$, none of whose roots are rational, but when we have found the four values of y we must substitute in $\{3(1)-(2)\}$ and find the corresponding value of x.

For the second part we find from (2), $x=(2y^2-22y+54)/y$. Substituting in (1), we get $7y^4-143y^3+1032y^2-2970y+2916=0$, one of whose roots is $y=3$, giving $x=2$.

14146. (R. F. Davis, M.A.)—If OP, OP′ are tangents from an external point O to a given circle, and any point T is taken upon the line joining their middle points R, R′ from which tangents are drawn whose chord of contact intersects RR′ in U, prove that OT, OU are at right angles.

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Solutions (1) *by* C. Joss, M.A.; (2) *by* H. D. Drury, M.A.

(1) The figure needs no explanation.

$CD.CO = CP^2 = CE.CT$;

therefore ODET is concyclic;

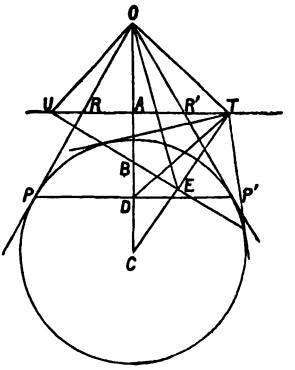
therefore $DOE = DTE$,

and $DBE = BOE + OEB$,

but DBE also = ATE, for BATE is concyclic; therefore

$$OEB = ATD = OTA$$

(for OA = AD); therefore OUET is concyclic.



(2) Let Q, Q′ be the points of contact of the tangents from T.

Since T is a point on the radical axis of the "point circle" O and the given circle, we have

$$TQ = TQ' = TO.$$

Therefore a circle described with T as centre and radius TO will pass through Q, Q′, and this circle will touch UO at O, for $UO^2 = UQ.UQ'$, since U is another point on the radical axis. Hence, &c.

14120. (Rev. T. Roach, M.A.)—Prove geometrically

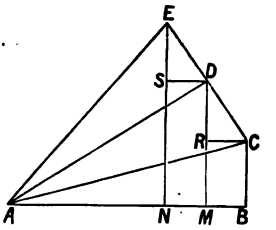
$$\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3.$$

Solution by the Proposer.

Draw AB = 7, and BC perpendicular = 1; then $CAB = \cot^{-1} 7$.

Make $CAD = \cot^{-1} 8$, and draw DM and CR perpendicular; then, by similar triangles, $CR = \frac{1}{8}$ and $DR = \frac{7}{8}$; therefore $AM = \frac{56}{8}$ and $DM = \frac{14}{8}$.

On AD make $DAE = \cot^{-1} 18$, and draw EN and DS perpendicular; then, by similar triangles, $SD = \frac{7}{4\,8}$ and $ES = \frac{56}{4\,8}$; therefore $AN = \frac{392}{4\,8}$ and $EN = \frac{224}{4\,8}$.



$EAN = \cot^{-1}(AN/EN) = \cot^{-1} 3$.

5665. (Professor Casey, M.A.)—Given the distance between the centres of two circles, the radius of one circle, and the length of an endless band that passes round the two circles; find the radius of the other circle.

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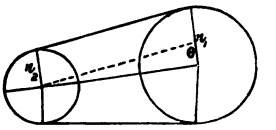
Solution by Professor SANJÁNA, M.A.

Let d = distance between centres, r_1, r_2 = radii of the circles, and $2l$ = length of endless band; then, if 2θ be the angle subtended at the centre of the larger circle by the free arc,



$$\cos\theta = (r_1 - r_2)/d;$$

hence $d\sin\theta = \{d^2 - (r_1 - r_2)^2\}^{\frac{1}{2}}$.

Now $2l = 2r_2\theta + 2r_1(\pi - \theta) + 2d\sin\theta$;

Therefore

$$l = r_2\cos^{-1}(r_1 - r_2)/d + r_1\{\pi - \cos^{-1}(r_1 - r_2)/d\} + \{d^2 - (r_1 - r_2)^2\}^{\frac{1}{2}};$$

whence one radius may be found when the other three quantities are given. When the band crosses between the circles the formula is

$$l = r_2(\pi - \theta) + r_1(\pi - \theta) + d\sin\theta.$$

14053. (Rev. W. A. WHITWORTH, M.A.)—Prove that Quest. 13630, Vol. LXIX., p. 47, is true if *any* square number be substituted for 4, *e.g.*,

$$1 - 9 + 9^2 - 9^3 + 9^4 = \square + \square = 73^2 + 24^2 = 63^2 + 44^2,$$

$$1 - 25 + 625 = \square + \square = 24^2 + 5^2;$$

and generally that

$$1 - a^2 + a^4 - a^6 + \ldots + a^{4r} = (1 - a^2 + a^4 - \ldots \pm a^{2r})^2 + (a - a^3 + \ldots \pm a^{2r-1})^2.$$

Solution by F. H. PEACHELL, B.A.; H. W. CURJEL, M.A.; *and* Lt.-Col. ALLAN CUNNINGHAM, R.E.

$$1 - a^2 + a^4 - a^6 + \ldots a^{4r} = 1 - a^2 + (a^2)^2 - (a^2)^3 + \ldots (a^2)^{2r}$$

$$= \frac{a^{4r+2} + 1}{a^2 + 1} = \left(\frac{a^{2r+2} \pm 1}{a^2 + 1}\right)^2 + \left\{\frac{a(a^{2r} \mp 1)}{a^2 + 1}\right\}^2 \quad \ldots\ldots\ldots\ldots (1)$$

$$= (1 - a^2 + a^4 - \ldots \pm a^{2r}) + (a - a^3 + \ldots \pm a^{2r-1})^2 \quad \ldots\ldots\ldots\ldots (2).$$

When r is odd, we take the bottom signs in (1) and (2), and the top signs when r is even.

14051. (R. TUCKER, M.A.)—P is a point on the ordinate of an ellipse produced. PK is the perpendicular from P on the polar of P. Find the equation to the circular locus of K, and the envelope of this locus as the ordinate varies in position. Interpret the result.

Solution by H. W. CURJEL, M.A.

Let Q be the pole of the ordinate. Then the polar of P (α, β) passes through Q $(a^2/\alpha, 0)$, and PK $\{(x - \alpha)/\alpha b^2 = (y - b)/a^2\beta\}$ passes through R $(c^2\alpha/a^2, 0)$. Therefore locus is a circle on QR as diameter. This clearly

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passes through the points of intersection of the ordinate and the ellipse and also through the foci on the y-axis, which therefore form the envelope of the circles. These circles are therefore coaxal and cut the circles through the foci on the x-axis orthogonally.

13998. (J. J. Barniville, M.A.)—Prove that the series 1, 1, 2, 3, 5, ... contains multiples of every number. Prove that u_{mn} is divisible by u_n. Can it be shown that u_n is prime when n is prime?
[See *Nature*, May, 1897.]

Solution by C. E. Bickmore, M.A.; *and* H. W. Curjel, M.A.

(1) $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\alpha + \beta = 1$, $\alpha\beta = +1$; u_n is divisible by an uneven prime p, if $(\alpha/\beta)^n \equiv 1 \pmod{p}$. Since $\alpha/\beta = -\frac{1}{2}(3 - \sqrt{5})$, this is satisfied by $n = p-1$, if 5 be a 2-ic residue of p, and by $n = p+1$, if 5 be a 2-ic non-residue of p. Thus

$$u_{10m} \equiv 0 \pmod{10m+1}, \quad u_{10m+8} \equiv 0 \pmod{10m+9},$$

$$u_{10m+4} \equiv 0 \pmod{10m+3}, \quad u_{10m+8} \equiv 0 \pmod{10m+7},$$

the modulus in each case being a prime. Similarly,

$$u_n \equiv 0 \pmod{p^k}, \quad \text{if } n = p^{k-1}(p \pm 1); \quad u_3 = 2, \; u_6 = 8;$$

therefore $u_n \equiv 0 \pmod{2^k}$, if $n = 3.2^{k-2}$, k being > 2.

(2) Since $(\alpha^{mn} - \beta^{mn})/(\alpha - \beta)$ is obviously divisible by $(\alpha^n - \beta^n)/(\alpha - \beta)$, u_{mn} is a multiple of u_n.

(3) If n be prime, u_n is not necessarily prime; thus u_{19} is a multiple of 37; but (noting that $u_{2n+1} = u_{n+1}^2 + u_n^2$), if n be a prime > 3, the prime factors of u_n are of the form $4mn+1$, if ending in 1 or 9, and of the form $(4m+2)n-1$ if ending in 3 or 7. As examples of the latter, a prime of form $60m+37$ always divides u_{30m+19}, and a prime of form $60m+13$ always divides u_{30m+7}.
Since one of the two numbers $10m+9$ and $10m+7$ is of the form $4M+1$, when they are both primes, one of them is a factor of u_{5m+4} and the other is not; *e.g.*, $m = 10$ gives 107 a factor of u_{108} but not of u_{54}; and u_{109} a factor of u_{54} as well as of u_{108}.

14005. (G. H. Hardy.)—Evaluate the definite integrals

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(p \tan^2 t)\, dt, \quad \int_0^{\frac{1}{2}\pi} \tan^{-1}(p \sin^2 t)\, dt,$$

$$\int_0^a \frac{(a_0 a_1 a_2 \ldots a_n \chi(\log x, 1)^n}{\sqrt{(\log a - \log x)}}\, dx, \quad \text{and} \quad \int_0^a \left(\frac{\sin}{\sinh}\right) \kappa \sqrt{(\log a - \log x)}\, dx.$$

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Solution by the PROPOSER.

(1) $$\int_0^{\frac{1}{2}\pi} \log(1+n\sin^2\theta)\,d\theta = \pi \log \tfrac{1}{2}\left\{1+\sqrt{(1+n)}\right\}.$$

Putting $n = ip$ and equating imaginary parts,

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(p\sin^2\theta)\,d\theta = \pi \tan^{-1}\frac{\sqrt{\left\{\sqrt{(1+p^2)}-1\right\}}}{\sqrt{2}+\sqrt{\left\{\sqrt{(1+p^2)}+1\right\}}}.$$

More generally, $$\int_0^{\frac{1}{2}\pi} \tan^{-1}(p\sin^2\theta/1+q\sin^2\theta)\,d\theta$$

$$= \pi \tan^{-1}\frac{\sqrt{\left[\sqrt{\left\{(1+q)^2+p^2\right\}}-(1+q)\right]}}{\sqrt{2}+\sqrt{\left[\sqrt{\left\{(1+q)^2+p^2\right\}}+(1+q)\right]}};$$

and, similarly, $$\int_0^{\frac{1}{2}\pi} \tan^{-1}(p\tan^2\theta)\,d\theta = \pi \cot^{-1}\left\{1+\sqrt{(2/p)}\right\}.$$

(2) It is easy to show that

$$\int_0^a \frac{dx}{\sqrt{(\log a-\log x)}} = a\sqrt{\pi} = \frac{2^{n+1}}{1.3\,..\,2n+1}\int_0^a (\log a-\log x)^{n+\frac{1}{2}}\,dx,$$

and so, expanding $(a_0 \ldots a_n \overline{\log x},\ 1)^n$ in powers of $\log a - \log x$, we find

$$\int_0^a \frac{(a_0 \ldots a_n \overline{\log x},\ 1)^n}{\sqrt{(\log a-\log x}}dx = a\sqrt{\pi}\left\{A_n - \tfrac{1}{2}nA_{n-1} + \frac{1.3}{2^2}\frac{n.n-1}{1.2}A_{n-2} - \ldots\right\},$$

where $A_0 = a_0$, $A_1 = a_0 \log a + a_1$, $A_2 = a_0(\log a)^2 + 2a_1 \log a + a_2$, &c.

Again, $$\int_0^a \sin\kappa\sqrt{(\log a-\log x)}\,dx$$

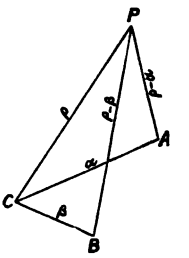
$$= \tfrac{1}{2}\kappa a\sqrt{\pi}\left\{1-(\tfrac{1}{4}\kappa)^2(1/1\,!) + (\tfrac{1}{4}\kappa)^4(1/2\,!) - \ldots\right\} = \tfrac{1}{2}\kappa a e^{-\frac{1}{4}\kappa^2}\sqrt{\pi},$$

and $$\int_0^a \sinh\kappa\sqrt{(\log a-\log x)}\,dx = \tfrac{1}{2}\kappa a e^{\frac{1}{4}\kappa^2}\sqrt{\pi}.$$

5366. (W. J. C. MILLER, B.A.)—Find the locus of a point whose distance from one of three given points is (1) an arithmetic mean, (2) a geometric mean, between its distances from the other two.

Solution by Rev. J. CULLEN.

Let A, B, and C be the given points, and ρ the vector to P, the variable point. Then we clearly have:



(1) $$2T\rho = T(\rho-\alpha) + T(\rho-\beta),$$

or $$4S^2\rho(\alpha+\beta+\rho) - 4(\alpha^2+\beta^2)S\rho(\alpha+\beta+\rho) + (\alpha^2+\beta^2)^2 - 4(\rho-\alpha)^2(\rho-\beta)^2 = 0;$$

(2) $$T^2\rho = T(\rho-\alpha)\times T(\rho-\beta),$$

or $$\rho^2\left[\alpha^2+\beta^2-2S(\alpha+\beta)\rho\right] + S\beta(\beta-2\rho)S\alpha(\alpha-2\rho) = 0,$$

two cubic surfaces, since in (1) ρ^4 disappears.

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Note on Mr. F. W. LAWRENCE's *paper on the* "Determination of certain Primes" (*Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 465).

By D. BIDDLE, M.R.C.S.

The above-named paper is devoted to the determination of five large numbers as primes, and I do not for one moment dispute the beauty or efficacy of the method employed. But I would observe that in each case something is admittedly known beforehand about the number to be investigated and the form of its possible factors, which greatly relieves the burden of the operator; and, such being the case, I think an easier method might have been adopted. The facts referred to are as follows:—

			Form of N's factors.
$N_1 =$	77,843,839,397,	a residuary factor of $10^{29}-1$,	$29m+1$;
$N_2 =$	182,521,213,001,	,, ,, $10^{25}-1$,	$25m+1$;
$N_3 =$	1,001,523,179,	,, ,, $3^{23}-1$,	$23m+1$;
$N_4 =$	12,014,633	norms of 7-ic trinomial integers,	$7m+1$.
$N_5 =$	13,477,969		

In each case we have

$$N = \Delta M + 1 = (\Delta P + 1)(\Delta Q + 1) = \Delta(\Delta PQ + P + Q) + 1,$$

and Δ is odd. Therefore P and Q are both even, and M is even. Let $M = \Delta R + a$; then $P + Q = \Delta r + a$, in which r and a are both even or both odd. $PQ = R - r$, and is divisible by 4. Half the sum of the factors of $N = \frac{1}{2}\Delta(P+Q) + 1 = \frac{1}{2}\Delta^2 r + \frac{1}{2}\Delta a + 1$, the trial values of which advance by uniform steps of $2\Delta^2$, because the value of r advances by 4. Moreover, we know that the half-sum of the factors is greater than S, the integral portion of $\sqrt{N}$. Therefore the commencing value of $r > (2S - \Delta a - 2)/\Delta^2$. Again, if N be known to have no factor less than G, then $r < (N + G^2)/(G\Delta^2)$. Consequently, $(S-G)^2/(4G\Delta^2)$ represents very closely the maximum number of trials requisite for factorizing N, or determining it a prime, under the foregoing conditions alone.

But we have a further guide in the table of bidigital endings of the possible values, given at the end of my Note on Quest. 14080 (Table IV., p. 98), where we find much to lighten our labour.

Taking Mr. LAWRENCE's $N_1 = 77,843,839,397$, we have $S = 279,005$, $\Delta = 29$, $M = 2,684,270,324$, $R = 92,561,045$, $a = 19$, whence r_1 might be 662. But $R - r \equiv 0 \pmod 4$, and $R \equiv 1 \pmod 4$; therefore $r \equiv 1 \pmod 4$, and $r_1 = 665$. Thus, taking H = half-sum of factors, $H_1 = 279,909$, agreeing with the table of bidigital endings, which, for $N = \ldots 97$, gives as finals 1 and 9, preceded by any digit as penultimate. Again, $N_d = 5$ = sum of digits of N; therefore $H_d = 3$, 6, or 9 (see Table II., p. 97). H_1 agrees with this also.

Now, in order to maintain the finals in the trial-values of H, we must add $2\Delta^2 (= 1682)$ and $8\Delta^2 (= 6728)$ alternately. But $(2\Delta^2)_d = 8$, and $(8\Delta^2)_d = 5$; therefore we must take $(2\Delta^2 + 8\Delta^2 + 2\Delta^2)$ and $(8\Delta^2 + 2\Delta^2 + 8\Delta^2)$ alternately, since these add 3 and 0 respectively to the sum of the digits. Thus, to H_1 we add alternately 10092 and 15138 in order to fulfil the requisite conditions. The tables for residue, index 10, show, as admitted by Mr. LAWRENCE, that G, the number below which there is no factor of

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N, is 100,000. Consequently $H < 439{,}219$, and this limit is passed after H_{13}. There are thus only 13 trial-values that need be considered in the case of N_1, and, even if they were much more numerous, the large majority are capable of being eliminated, without being previously set down, by a very simple process suggested by Mr. CULLEN's Note to Quest. 14080, p. 99. We take a few of the earlier primes as moduli and proceed as follows:—

	(mod 7),	(mod 11),	(mod 13),	(mod 17),	(mod 19)
$N \equiv$	$+2$,	$+5$,	-1	$+5$	-4.
$H \equiv$	$\pm 2, 3$;	$\pm 3, 4, 5$;	$\pm 0, 3, 4, 5$;	$\pm 1, 2, 3, 8$;	$\pm 0, 1, 4, 8, 9$
$279{,}909 \equiv$	$(0)\ H_1$	$+3$	$(+6)\ H_1$	$(+4)\ H_1$	$+1$
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$290{,}001 \equiv$	-2	-3	-3	-2	$+4$
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$305{,}139 \equiv$	$+2$	$(-1)\ H_3$	$+3$	$(+6)\ H_3$	-1
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$315{,}231 \equiv$	$(0)\ H_4$	$+4$	$(-6)\ H_4$	$(0)\ H_4$	$(+2)\ H_4$
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$330{,}369 \equiv$	-3	-5	0	$+8$	$(-3)\ H_5$
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$340{,}461 \equiv$	$+2$	$(0)\ H_6$	$+4$	$+2$	0
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$355{,}599 \equiv$	$(-1)\ H_7$	$(+2)\ H_7$	-3	$(-7)\ H_7$	-5
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$365{,}691 \equiv$	-3	-4	$(+1)\ H_8$	$(+4)\ H_8$	$(-2)\ H_8$
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$380{,}829 \equiv$	$(+1)\ H_9$	$(-2)\ H_9$	$(-6)\ H_9$	$(-5)\ H_9$	$(-7)\ H_9$
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$390{,}921 \equiv$	$(-1)\ H_{10}$	$+3$	$(-2)\ H_{10}$	$(+6)\ H_{10}$	-4
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$406{,}059 \equiv$	$+3$	$+5$	$+4$	-3	-9
$10{,}092 \equiv$	-2	$+5$	$+4$	-6	$+3$
$416{,}151 \equiv$	$(+1)\ H_{12}$	$(-1)\ H_{12}$	5	$+8$	$(-6)\ H_1$
$15{,}138 \equiv$	-3	$+2$	$+6$	$+8$	-5
$431{,}289 \equiv$	-2	$(+1)\ H_{13}$	$(+1)\ H_{13}$	-1	$+8$

The residues excluded under each modulus are inclosed in parentheses, and have the number of the trial-value H_n placed against each. Only two out of the thirteen are not eliminated, namely, H_2 and H_{11}. On squaring these and subtracting N, the remainders are neither of them square; therefore N_1 is determined a prime.

Observe that after the commencement of the process sketched above, the successive residues in each column are obtained without recourse to the values given at the left hand, which are set down for satisfaction only.

In the case of N_2, we are still aided by the fact that $G = 100{,}000$, so that, although Δ is less than in N_1, the labour is not much greater.

In the case of N_3, we are not aided by G; but, as the number itself consists of 10 figures only, the process of elimination above described will not be unduly prolonged; and the same remarks apply to N_4, N_5.

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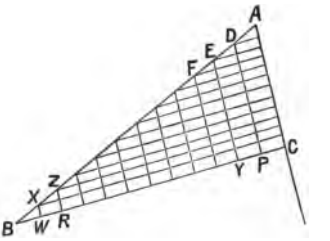
[Where the number of trial-values exceeds $2p$, in which p is a particular modulus 7, 11, 13, &c., the excluded values are in the same relative position in each succeeding set of $2p$ values. Therefore, having excluded from the first set, we can easily exclude from the rest by adding multiples of $2p$ to the numbered positions of those excluded from the first set. Then, the values *not* excluded, in the whole series, being also numbered, can readily be found by adding to H_1, $\frac{1}{2}(n-1)(a+b)+0$ or $\frac{1}{2}(n-2)(a+b)+a$, according as n is odd or even; a, b being the alternate amounts added.]

14064. (I. Arnold.)—Through a given point P between the legs of a given angle A, to draw a line BPC so that BP may be m times PC, and then show (without duplicate ratio) that equiangular triangles are as the squares of the homologous sides, using Book I. only.

Solution by Professor T. Savage; F. H. Peachell, B.A.; *and others.*



Through P draw PD parallel to AC. From DB cut off m parts, DE, EF, ..., XB, each equal to AD. Join BP, and produce to C. Through E, F, ..., X, draw parallels to AC, BC.

By a well-known deduction, it is at once shown that

$$BW = WR = \ldots = YP = PC;$$

therefore BP is m times PC.

Again, it is obvious that the triangle BDP is composed of $(1+2+3+\ldots+m-1)$ parallelograms of area XR and m triangles of area BWX—that is, of $m(m-1)+m$ triangles BWX, or of m^2 triangles BWX. Similarly, the triangle BAC is composed of $(m+1)^2$ triangles BWX. Hence

$$\triangle BDP : \triangle BAC :: m^2 : (m+1)^2.$$

14073. (Prof. S. Sircom, M.A.)—If n is a positive integer, prove that

$$\int_0^\infty \frac{\sin^n\theta}{\theta^n}d\theta = \frac{1}{(n-1)!\,2^n}\left\{n^{n-1} - n(n-2)^{n-1} + \frac{n(n-1)}{1.2}(n-4)^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}(n-6)^{n-1} + \ldots\right\}\frac{\pi}{2}.$$

[I have only seen this question worked for low values of n.]

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Solution by Professor K. J. SANJÁNA, M.A.

It may be proved that, when n is *odd*,

$$\sin^n\theta = \frac{1}{2^{n-1}(-1)^{\frac{1}{2}(n-1)}}\Big[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta$$
$$-\ldots+(-1)^{\frac{1}{2}(n-1)}\frac{n!}{\{\frac{1}{2}(n-1)\}!\{\frac{1}{2}(n+1)\}!}\sin\theta\Big].$$

Hence, multiplying each term by θ^{-n}, and integrating, we get

$$\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta$$
$$= \frac{1}{2^{n-1}(-1)^{\frac{1}{2}(n-1)}}\Big\{\int_0^\infty \frac{\sin n\theta}{\theta^n}\,d\theta - n\int_0^\infty \frac{\sin(n-2)\theta}{\theta^n}\,d\theta+\ldots\Big\}$$
$$= \frac{1}{2^{n-1}(-1)^{\frac{1}{2}(n-1)}}\Big\{n^{n-1}-n(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}-\ldots\Big\}$$
$$\times\frac{\pi}{\Gamma(n).2\sin\frac{1}{2}n\pi}$$
$$= \frac{1}{2^n(n-1)!}\Big\{n^{n-1}-n(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}(n-6)^{n-1}+\ldots\Big\}\pi.$$

It may be proved that, when n is *even*,

$$\sin^{n-1}\theta\cos\theta$$
$$= \frac{1}{2^{n-1}(-1)^{\frac{1}{2}(n-2)}}\Big[\sin n\theta - 1(n-2)\sin(n-2)\theta + \frac{(n-1)(n-4)}{1.2}\sin(n-4)\theta$$
$$-\ldots+(-1)^{\frac{1}{2}(n-2)}\frac{(n-1)!\,2}{\{\frac{1}{2}(n-2)\}!\{\frac{1}{2}(n+2)\}!}\sin 2\theta\Big].$$

Hence, in this case, $\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta$

$$= -\frac{1}{n-1}\int_0^\infty \sin^n\theta\,\frac{d\theta^{-(n-1)}}{d\theta}\,d\theta$$
$$= -\frac{1}{n-1}\Big[\frac{\sin^n\theta}{\theta^{n-1}}\Big]_0^\infty + \frac{n}{n-1}\int_0^\infty \frac{\sin^{n-1}\theta\cos\theta}{\theta^{n-1}}\,d\theta$$
$$= \frac{n}{n-1}\cdot\frac{1}{2^{n-1}(-1)^{\frac{1}{2}(n-2)}}\Big\{\int_0^\infty \frac{\sin n\theta}{\theta^{n-1}}d\theta-(n-2)\int_0^\infty \frac{\sin(n-2)\theta}{\theta^{n-1}}d\theta+\ldots\Big\}$$
$$= \frac{n}{n-1}\cdot\frac{1}{2_n{}^{-1}(-1)^{\frac{1}{2}(n-2)}}\Big\{n^{n-2}-(n-2)(n-2)^{n-2}+\frac{(n-1)(n-4)}{1.2}(n-4)^{n-2}$$
$$-\frac{(n-1)(n-2)(n-6)}{1.2.3}(n-6)^{n-2}+\ldots\Big\}\frac{\pi}{\Gamma(n-1)\,2\sin\frac{1}{2}(n-1)\pi}$$
$$= \frac{1}{2^n(n-1)!}\Big\{n^{n-1}-n(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}(n-6)^{n-2}+\ldots\Big\}\pi.$$

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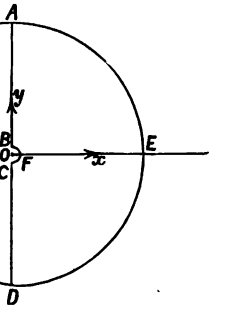
Thus the final result is the same in both cases. It should be remembered that $\sin \infty = 0$, and $\Gamma(m) \int_0^\infty \frac{\sin n\theta}{\theta^m}\, d\theta = \pi n^{m-1}/2 \sin \frac{1}{2}m\pi$.

14155. (G. H. Hardy.)—Prove, by contour integration, or otherwise, that

$$\int_0^\infty e^{n\cos\alpha x} \sin(\beta x + n \sin \alpha x)\,(dx)/x = \tfrac{1}{2}\pi e^n.$$

What conditions, if any, must α, β, n satisfy?

Solution by G. D. Muggeridge.



Consider $\int \frac{\epsilon^{(ne^{-\alpha z} - \beta z)}}{z}\, dz \qquad (z \equiv x + iy)$

taken round the contour ABFCDE, AED being a semicircle of infinite radius, and BFC one of radius δ, ultimately zero.

Now, if β *be positive* and α *be positive;* or, if β *be positive* and α *be negative* and n *negative*, the subject of integration is everywhere finite within the contour. Hence the value of the integral is zero.

Therefore $$0 = \int_-^{+\infty} \frac{\epsilon^{(ne^{-\alpha iy} - \beta iy)}}{y}\, dy + \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{\epsilon^{(ne^{-\alpha\delta e^{i\phi}} - \beta\delta e^{i\phi})}}{\delta\epsilon^{i\phi}}\, i\delta\epsilon^{i\phi}\, d\phi,$$

where, by $$\int_{-\infty}^{+\infty} \frac{\epsilon^{ne^{-\alpha iy} - \beta iy}}{y}\, dy,$$

we denote the *principal value* of that integral.

Hence $$\int_{-\infty}^{+\infty} \frac{\epsilon^{n\cos\alpha y - (\beta y + \sin\alpha y)i}}{y}\, dy = -\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} i\epsilon^n\, d\phi = -i\pi\epsilon^n;$$

therefore $$\int_{-\infty}^{+\infty} \frac{\epsilon^{n\cos\alpha y}\{\cos(\beta y + \sin\alpha y) - i\sin(\beta y + \sin\alpha y)\}}{y}\, dy = -i\pi\epsilon^n;$$

therefore $$\pi\epsilon^n = \int_{-\infty}^{+\infty} \frac{\epsilon^{n\cos\alpha y}\sin(\beta y + \sin\alpha y)}{y}\, dy$$

$$= 2\int_0^\infty \frac{\epsilon^{n\cos\alpha y}\sin(\beta y + \sin\alpha y)}{y}\, dy.$$

[Other possible cases for which β is negative may be similarly considered.]

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5377. (Rev. A. F. TORRY, M.A.)—Prove that the chord which joins the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $l\alpha^{\frac{1}{2}} + m\beta^{\frac{1}{2}} + n\gamma^{\frac{1}{2}} = 0$ is parallel to $$\frac{l\alpha}{\alpha_1^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}} + \frac{m\beta}{\beta_1^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}} + \frac{n\gamma}{\gamma_1^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}} = 0.$$

Solution by the PROPOSER.

The equation to chord is

$$\frac{l(\alpha-\alpha_1)}{\alpha_1^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}} + \frac{m(\beta-\beta_1)}{\beta_1^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}} + \frac{n(\gamma-\gamma_1)}{\gamma_1^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}} = 0,$$

or $$\frac{l\alpha}{\alpha_1^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}} + \frac{m\beta}{\beta_1^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}} + \frac{n\gamma}{\gamma_1^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}} = \frac{l\alpha_1}{\alpha_1^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}} + \frac{m\beta_1}{\beta_1^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}} + \frac{n\gamma_1}{\gamma_1^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}};$$

and the result immediately follows.

5495. (Professor KELLAND, F.R.S.) — Find, by quaternions, the locus of the intersections of perpendiculars from a given point on planes which pass through a given point.

Solution by Rev. J. CULLEN.

Let ρ be the vector to the foot of the perpendicular on one of the planes, and α the vector to the given point through which this plane passes. Then, since $\rho - \alpha$ lies in the plane, we have

$$S\rho(\rho-\alpha) = 0, \quad \text{or} \quad \rho^2 - S\alpha\rho = 0.$$

Hence the locus is a sphere whose centre is the mid-point of α and radius $= T(\frac{1}{2}\alpha)$.

5645. (H. MACCOLL, B.A.)—If, from the centre C of a square, straight lines CP_1, CP_2, CP_3, ... are drawn at equal angular intervals to meet the perimeter in the points P_1, P_2, P_3, ..., show (1) that (the number of these lines being increased without limit) the average area of the circle of which the variable CP is the radius is equal to the area of the given square; and (2) that this is also true if we substitute equilateral and equiangular polygon for the word square.

Solution by H. W. CURJEL, M.A.

$$\text{Average area} = \int_0^{2\pi} \pi r^2 \, dr \Big/ \int_0^{2\pi} d\theta = \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta = \text{area of figure.}$$

This is clearly true in any closed curve if C is such a point that the join of C to any point on the perimeter does not cut the perimeter between C and P.

[See solutions of 3292, Vol. XV., p. 104, and of 3486, Vol. XVI., p. 112, in which latter the same principle is extended to solids and "any point

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C in space," by the same PROPOSER, who has found his original Question (28 years old) in the following lines of verse:—

Some shapeless solids lie about,
No matter where they be;
Within such solid, or without,
Let's take a centre C.
From centre C, in countless hosts,
Let random radii run,
And meet a surface, each at P,
Or, may be, meet with none.
Those shapeless solids, far and near,
Their total prove to be
The average volume of the sphere
Whose radius is CP.
The sphere, beware, is positive,
If *out* at P they fly,
But, changing sign, 'tis negative,
If *entrance* there you spy.
One caution more and I have done:
The sphere is naught, if P there's none.]

5603. (C. LEUDESDORF, M.A.)—Within a given circle a point is taken at random, and with this as centre and any radius taken at random a circle is described so as to lie wholly within the first. If the first circle be now reciprocated with regard to the second, find the average area of the resulting ellipse.

Solution by H. W. CURJEL, M.A.

If r is the radius of the variable circle and x the distance of its centre from that of the fixed circle, the area of the ellipse

$$= \pi \frac{r^4}{a} \frac{a}{a^2 - x^2}.$$

Therefore average area $= \left(\pi \int_0^a \int_0^{a-x} \frac{r^4 dr}{(a+x)(a-x)^2} x\, dx \right) \Big/ \int_0^a x\, dx$

$$= \frac{2\pi}{5a^2} \int_0^a \frac{(a-x)^5}{(a+x)(a-x)^2} x\, dx = \frac{2\pi a^2}{5} \left\{ 7\tfrac{1}{4} - 10 \log 2 \right\}.$$

5589. (Rev. W. A. WHITWORTH, M.A.)—Two persons are known to have passed over the same route in opposite directions within a period of time $m+n+r$, the one occupying time m and the other time n. Find the chance that they will have met.

Solution by H. W. CURJEL, M.A.

If they are both on the route at any instant, they must meet. If A (who occupies time m) starts at time x, they will not meet if B

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starts after time $m+x$ or before the time $x-n$. Therefore chance that they will not meet

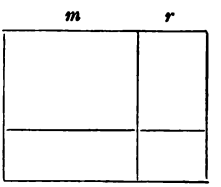
$$= \frac{1}{n+r}\left\{\int_0^r \frac{r-x}{m+r}\,dx + \int_n^{n+r} \frac{x-n}{m+r}\,dx\right\}$$

$$= \frac{1}{(n+r)(m+r)}\left\{\int_0^r (r-x)\,dx + \int_0^r x\,dx\right\} = \frac{r^2}{(n+r)(m+r)}.$$

Therefore chance that they will meet is $\dfrac{r(m+n)+mn}{(n+r)(m+r)}$.

[The Proposer says: "This Question is quite obsolete, having become bookwork long ago. It must be twenty-five years or more since I sent it in. The only interest of the Question was in its solution by a geometrical figure. Possible cases cover area $(m+r)(n+r)$; unfavourable cover area r^2; therefore chance

$$= 1-r^2/\{(m+r)(n+r)\}."]$$



14154. (H. J. Woodall, A.R.C.S.) — Find the value of x in the following table:—

0	·493 107 418 043 066 689
1	·502 687 750 553 717 016
2	·512 251 521 229 044 033
3	·521 798 422 844 633 606
4	·531 328 149 182 931 742
5	·540 840 395 051 310 490
6	·550 334 856 300 060 834
7	x
8	·569 269 213 661 871 302
9	·578 708 506 850 997 002

Solution by H. Fortey.

Call the terms in the table $u_0, u_1, \ldots, u_9$. Then, assuming $\Delta^9 u_0 = 0$, we have

$$u_9 - 9u_8 + \frac{9.8}{1.2}u_7 - \ldots - u_0 = 0,$$

or $u_9 - u_0 - 9(u_8 - u_1) + 36(u_7 - u_2) - 84(u_6 - u_3) + 126(u_5 - u_4) = 0.$

Therefore $u_7 = u_2 + \frac{1}{126}\{-(u_9 - u_0) + 9(u_8 - u_1) + 84(u_6 - u_3) - 126(u_5 - u_4)\}$;

and, substituting the known values of $u_0, u_1, \ldots$, we get

$$u_7 \text{ or } x = ·559\ 811\ 229\ 840\ 311\ 343.$$

Inserting this value in the table and taking differences, I find $\Delta^9 u_0 = -5/10^{18}$, so that x must be correct nearly to the last figure.

[The Proposer remarks: "The numbers in the Question are the values of Si a for $a = ·50, 1, 2, 3, 4, 5, 6, 8, 9$, given by Dr. Glaisher, *Phil. Trans.*, Vol. clx. (where "Si" stands for "sine-integral"). The tabulated value for 7 differs from the calculated value only in the last place giving 8 instead of 3."]

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13651. (R. F. Muirhead.)—In the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{matrix}$$

there are 40 determinants which can be formed by suppressing 2 columns and 1 row. Show that six of these can be chosen so that their vanishing involves the vanishing of all the others. Extend this to an array of p columns and q rows, and determinants of the order r, where r is less than either p or q.

Solution by Professor Nanson.

If all the determinants of the second order which can be formed from the given array vanish, then so do all those of the third order, for each of them is a linear function of those of the second order. But, if not, let (a_1b_2) be one of the determinants of the second order which do not vanish; then, if the six determinants

$$\begin{matrix} (a_1b_2c_3), & (a_1b_2d_3), & (a_1b_2e_3), \\ (a_1b_2c_4), & (a_1b_2d_4), & (a_1b_2e_4) \end{matrix}$$

all vanish, each of the forty determinants $(x\ ,\ y_\beta,\ z_\gamma)$ vanishes where x, y, z are any three of the five a, b, c, d, e, and α, β, γ are any three of the four 1, 2, 3, 4.

For, let a_x denote $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, and similarly for b_x, &c.; then the five planes $a_x = 0$, $b_x = 0$, $c_x = 0$, $d_x = 0$, $e_x = 0$ meet the face $x_3 = 0$ of the tetrahedron of reference in the same point P_3 and the face $x_4 = 0$ in the same point P_4. These points P_3, P_4 cannot coincide unless each of them is on the edge $x_3 = 0$, $x_4 = 0$, and then only when the planes a_x, b_x meet this edge in the same point. But since $(a_1b_2) \neq 0$, the planes a_x, b_x meet this edge in distinct points. Thus the points P_3, P_4 are distinct, and the five planes have P_3, P_4 for a common line of intersection, and therefore the traces of any three of them on any one of the four faces x_1, x_2, x_3, x_4 are concurrent, and therefore $(x_\alpha y_\beta z_\gamma) = 0$.

The extension of the theorem is as follows:—

From a rectangular array (a_{xy}) where $x = 1, 2, \ldots p$; $y = 1, 2, \ldots q$; can be formed ${}_pC_r \cdot {}_qC_r$ determinants of order r, and these all vanish if $(p-r+1)(q-r+1)$ of them, properly chosen, vanish.

Let $\Delta_{\theta\phi} = |\ a_{\theta\phi}\ |$, where θ, ϕ are $(r-1)$-ads from $1, 2, \ldots, p$; $1, 2, \ldots, q$, respectively be a determinant of order $r-1$, which does not vanish. Then, if the $(p-r+1)(q-r+1)$ determinants of order r

$$\begin{vmatrix} (a_{\theta\eta}) & (a_{\theta\eta}) \\ (a_{\xi\phi}) & a_{\xi\eta} \end{vmatrix}$$

all vanish, where ξ is any one of the numbers $1, 2, \ldots p$ not included in the set θ, and η is any one of the numbers $1, 2, \ldots q$ not included in the set ϕ, then all the ${}_pC_r \cdot {}_qC_r$ determinants of order r vanish.

For, if D_{xy} denote the determinant

$$\begin{vmatrix} (a_{\theta\phi}) & (a_{\theta y}) \\ (a_{x\phi}) & a_{xy} \end{vmatrix},$$

then D_{xy} vanishes identically if x be any one of the set θ, or if y be any one of the set ϕ, for then two rows or two columns are identical. Also

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D_{xy} vanishes by hypothesis when x, y are not included respectively in the sets θ, ϕ. Hence D_{xy} vanishes for all values of x, y.

Now, if $-b_{y\phi}$ be the co-factor of $a_{x\phi}$ in D_{xy}, we have, since $D_{xy} = 0$,

$$da_{xy} = \Sigma_\phi a_{x\phi} b_{y\phi}.$$

Hence the array (da_{xy}) is compounded of the deficient arrays $(a_{x\phi})$, $(b_{y\phi})$, which have $r-1$ columns, and have respectively p, q rows. Hence, if λ, μ be any r-ads from $1, 2, \ldots, p$; $1, 2, \ldots, q$, respectively, the determinant $|\, da_{\lambda\mu} \,|$ vanishes. Thus $d^r \,|\, a_{\lambda\mu} \,| = 0$,

and, since $d \neq 0$, we get $|\, a_{\lambda\mu} \,| = 0$, the result to be proved.

The same method may be used to express the determinant $|\, a_{\lambda\mu} \,|$ in terms of the determinant D_{xy}. For we have

$$da_{xy} - D_{xy} = \Sigma_\phi a_{x\phi} b_{y\phi},$$

and therefore, as before, $|\, da_{\lambda\mu} - D_{\lambda\mu} \,| = 0$.

On expansion, we obtain for $d^r \,|\, a_{\lambda\mu} \,|$ an expression each term of which has one or more of the determinants $D_{\lambda\mu}$ as a factor. We have, in fact,

$$d^r \,|\, a_{\lambda\mu} \,| = d^{r-1}\Delta_1 - d^{r-2}\Delta_2 + \ldots + (-1)^r d\Delta_{r-1} + (-1)^{r-1} \,|\, D_{\lambda\mu} \,|\,,$$

where Δ_1, Δ_2, ..., Δ_{r-1} are the invariants intermediate to the discriminants of the quadrics $\quad \Sigma a_{\lambda\mu} X_\lambda X_\mu, \quad \Sigma D_{\lambda\mu} X_\lambda X_\mu.$

[There is another solution of this Question in Vol. LXIX., p. 52, which, however, is incomplete, and breaks down in particular cases, namely, when the quantities K, L, M and F, G, H are zero, though it is quite correct if such special cases be excluded. Professor NANSON's solution did not arrive until the other was published, concerning which latter he now says: "If $a_2 = b_2 = c_2 = d_2 = e_2 = 0$, then the six third-order determinants selected in the solution all vanish, but the determinant $(a_1 b_3 c_4)$ does not vanish."]

Note géométrique relative à la Quest. 14071, *par* Prof. A. DROZ-FARNY.

L'hexagone BCAA'PB' étant inscrit dans la conique, les trois points d'intersection $\begin{matrix} BC \\ A'P \end{matrix} \Big| \alpha, \quad \begin{matrix} CA \\ PB' \end{matrix} \Big| \beta$ et $\begin{matrix} AA' \\ BB' \end{matrix} \Big| Q$ sont en ligne droite.

L'hexagone CABB'PC' étant aussi inscrit, le point Q' d'intersection des côtés BB' et CC' est aussi sur la droite u. Les deux points Q et Q' coincident car BB' ne rencontre u qu'en un seul point. Les points P sur la conique et Q sur la droite u se correspondent homographiquement, donc, d'après SCHRÖTER ("Ueber die Erzeugnisse krummer projectivischer Gebilde," *Journal de Crelle*, Vol. LIV.), la droite PQ enveloppe une quartique de troisième classe, doublement tangente à u et triplement tangente à la conique.

Supposons comme conique le cercle circonscrit au triangle ABC, et la transversale u à l'infini. Les droites AA', BB', CC' deviennent les conjuguées isogonales des droites PA, PB, PC; elles sont parallèles à l'axe de la parabole de foyer P et inscrite au triangle ABC.

Comme il est facile de le prouver, l'enveloppe de PQ touche dans ce cas la transversale u aux ombilics du plan, d'où le théorème connu: Les axes des paraboles inscrites à un triangle enveloppent une hypocycloïde de STEINER.

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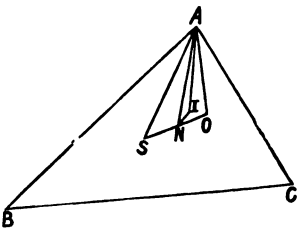
Note on Quest. 14014 *by the* PROPOSER.

As to the second part of the Question (see Vol. LXXI., p. 77), let S, O, N, I be the circum-, ortho-, nine-point-, and in-centres; then AN bisects SO, and AI bisects SAO. Thus, by the first part of the Question, $AN \cos NAI = \frac{1}{2}(AS + AO)\cos\frac{1}{2}SAO$;



also $AN \cos NAI = (AN^2 + AI^2 - IN^2)/2AI$.

Combining these, we have

$$IN^2 = AN^2 + AI^2 - AI\,(AS + AO)\cos\tfrac{1}{2}SAO.$$

Now $AS = R$, $AO = 2R\cos A$, $AI = 4R\sin\frac{1}{2}B\sin\frac{1}{2}C$,

$$SAO = \tfrac{1}{2}(C - B);$$

also, as $$SO = R\sqrt{(1 - 8\cos A\cos B\cos C)}$$

and $$4AN^2 + SO^2 = 2AS^2 + 2AO^2,$$

therefore $$AN = \tfrac{1}{2}R\sqrt{(1 + 8\cos A\sin B\sin C)}.$$

Hence
$$\begin{aligned} IN^2 &= \tfrac{1}{4}R^2 + 2R^2\cos A\sin B\sin C + 16R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C \\ &\quad - 4R^2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(C - B) \\ &\quad - 8R^2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos A\cos\tfrac{1}{2}(C - B) \\ &= R^2\{\tfrac{1}{4} + 2\cos A\sin B\sin C + 16\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C \\ &\quad - 4\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C - 4\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C \\ &\quad - 2\cos A\sin B\sin C - 8\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\cos A\} \\ &= R^2\{\tfrac{1}{4} + 4\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C(\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}B\cos\tfrac{1}{2}C) \\ &\quad + 8\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C(1 - \cos A)\} \\ &= R^2\{\tfrac{1}{4} - 4\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\sin\tfrac{1}{2}A + 16\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\sin^2\tfrac{1}{2}A\} \\ &= (R^2 - 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C)^2; \end{aligned}$$

so that $$IN = \tfrac{1}{2}R - r.$$

This proves the theorem for the incircles. Similar reasoning applies to the excircles.

Note on MERSENNE-FERMAT'S *Problem.* *By* ROBERT RAWSON, J.P. for Hants.

MERSENNE asked FERMAT whether N = 100895598169 was a prime. FERMAT, on April 7, 1643, said it was not a prime, but was the product of 898423 by 112303, both of which were primes.

The discovery of the method by which FERMAT arrived at this result has been one of the puzzles of higher arithmetic. See Note on this problem in the *Messenger of Mathematics*, Vol. XXII., p. 82, where

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W. W. Rouse Ball has also given a solution of this puzzle depending upon a theorem by Mr. Birch.

Mr. Birch's theorem is to factorize all numbers of the form $N = (x^2-1)/y$, where x and y are indeterminate integers—when N is a given number the solution of this problem seems to be as difficult as the original problem by Mersenne. It is quite correct that, if $x = 1796847$ and $y = 32$, then will $N = 100895598169$, but W. W. Rouse Ball gives little, if any, information with respect to the method by which these values have been obtained. May I ask if the crux of the puzzle be still in the solution of $N = (x^2-1)/y$, and not in Mr. Birch's theorem? The following seems to point to this conclusion :—

$$N = (1796847^2-1)/32 = (1796848 \cdot 1796846)/32$$
$$= (16 \cdot 112303 \cdot 2 \cdot 898423)/32 = 112303 \cdot 898423.$$

Mr. H. J. Woodall, A.R.C.S., set this number for factorization in the *Educational Times* as Quest. 13975, and it has been factorized by the application of Mr. Biddle's method of triplets (see *Math. Reprint*, Vol. lxx., p. 123). It is possible that Mr. Biddle's method may claim some kinship to the method of Fermat.

Another solution of Mersenne's problem has been in my possession for some time founded upon the theory of algebraical equations.

The judgment of others will best determine whether it can claim any relationship to Fermat's method, and thereby throw a little light on the intolerable puzzle. The solution is as follows :—

Let
$$N = 100895598169$$
$$= 10089559 \cdot 10^4 + 81 \cdot 10^2 + 69$$
$$= (10089559-n) \cdot 10^4 + (10^2n + 81) \cdot 10^2 + 69 \quad \ldots\ldots (1).$$

Equation (1) is solvable in integers by a quadratic if (n), an integer, is determined from

$$(10^2n + 81)^2 - 276\,(10089559-n) = \square \quad \ldots\ldots\ldots\ldots\ldots (2).$$

It is readily seen that (n) must be greater than 526 to make (2) positive. Take $n = 527$; then (1) becomes

$$N - 10089032 \cdot 10^4 + 52781 \cdot 10^2 + 69 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

But $52781^2 = 2785833961$ and $276 \cdot 10089032 = 2784572832$;

therefore $52781^2 - 276 \cdot 10089032 = 1 \cdot 261129 = 1123^2$.

Hence (3) becomes $N = (8984 \cdot 10^2 + 23)(1123 \cdot 10^2 + 3)$
$= 898423 \cdot 112303.$

The prime properties of these numbers might have been obtained by the ordinary rule without much trouble.

Additions to "Method of Factorization by H. J. Woodall, A.R.C.S.," (*Reprint*, Vol. lxx., pages 68–71.)

By extending the tables to include residues of powers of 2, 3, 5, 10 for powers of primes as well as primes themselves, it will be possible to show the existence of repeated factors. (This was pointed out to the

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author by a friend who possessed the whole method in its entirety long before the author had committed it to writing.)

Having completed such tables up to mod <1000, we add here the results as they affect the tables of factors given, Vol. LXX., pages 70 and 71. (N.B.—Where a power is given, it must be taken to *supersede* that given in the tables.)

Number.	Additional Factors.	Number.	Additional Factors.
$10^{15}-1$	3^3	$10^{15}+18$	23^2
-3	599	$+26$	3^4
-6	7^2	$+30$	613
-9	653	$+41$	577
-13	709	$+43$	7^2
-27	19^2	$+53$	3^3
-28	3^3	$+64$	727×967
-30	839	$+66$	13^2
-32	811	$+68$	857
-43	11^2	$+73$	17^2
-54	773	$+78$	11^2
-55	$3^4 \times 7^2 \times 521$	$+80$	3^3
-58	677	$+81$	619
-64	641	$+84$	757
-68	919	$+90$	541
-69	883	$+92$	7^3
-70	607	$+94$	571
-73	797	$+99$	587
-80	673	$+100$	859
-82	3^3	$+107$	3^6
-85	769	$+112$	647
-103	13^2	$+125$	937
-104	7^2	$+127$	659
-109	3^3	$+129$	929
-124	877	$+133$	977
-128	569	$+134$	3^3
-135	557	$+138$	701
-136	3^5	$+141$	7^2
-146	683	$+150$	617
-153	7^2	$+160$	509
-163	3^3	$+161$	$3^3 \times 953$
-164	11^2	$+173$	631
-177	523	$+185$	661
-185	601	$+188$	3^4
-190	3^3	$+190$	7^2
-202	7^2	$+192$	643
-206	733	$+199$	11^2
-216	17^2	$+203$	563
-217	3^4	$+213$	29^2
		$+215$	3^3

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Note on Quest. 13939. *By* C. E. HILLYER, M.A.

Since the publication of my Solution of this Question (see Vol. LXXI., p. 59) I have received from Dr. T. MUIR, of Capetown, an extract from a paper communicated by him to the *Edinburgh Mathematical Society* (Session 1884–85) on "Theorems connected with three mutually Tangent Circles," which deals very completely with the subject, giving a formula of which mine is a particular case.

13959. (Professor F. ROGEL.)—Prove that, if

$$(1)\quad K_p = q + 2\sum_{\nu=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}\nu}}{(\nu+1)!}\, B\left(\frac{1+q}{2}, \nu+1\right) \cdot \left(\frac{2\pi}{p}\right)^{\nu},$$

where $$q = (p-1)!, \quad p > 3,$$

and B denotes the "Bernoullian function" of the $(\nu+1)$th order;

$$(2)\quad L_p = -1 + \sum_{\nu=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu+2)}}{\nu!}\, E(r, \nu)\left(\frac{\pi}{2p}\right)^{\nu},$$

where $$r = \left[1 \cdot 3 \cdot 5 \ldots (p-2)\right]^2, \quad p > 3,$$

and E denotes the "Eulerian function" of the first species of the νth order. [F. ROGEL, "Theorie der Euler'schen Function," *Sitzungsbericht der kgl. böhmischen Gesellschaft der Wissenschaften*, Prag, 1893, XXIII.];

$$(3)\quad M_p = (-1)^{\frac{1}{2}(p-1)} \sum_{\nu=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu-1)}}{\nu!}\, E(q, \nu)\left(\frac{\pi}{2p}\right)^{\nu}, \quad p > 3;$$

$$(4)\quad N_p = \sum_{\nu=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu-1)}}{\nu!}\, E(s, \nu)\left(\frac{\pi}{2p}\right)^{\nu},$$

where $$s = \left[\left(\frac{p-1}{2}\right)!\right]^2, \quad p > 3,$$

each of these series K_p, L_p, M_p, $N_p = 1$ if p be a prime number, and $= 0$ if p be complex.

Solution by the PROPOSER.

1. We have, according to WILSON's theorem,

$$(p-1)! \equiv q = np - 1,$$

$$K_p \equiv \frac{\sin q\pi/p}{\sin \pi/p} = \begin{cases} 1, & \text{if } p \text{ be an odd prime number,} \\ 0, & \text{if } p \text{ be compound;} \end{cases}$$

or, setting $i(2\pi/p) = v$,

$$K_p = \frac{e^{\frac{1}{2}(1+q)v}}{e^v - 1} - \frac{e^{\frac{1}{2}(1-q)v}}{e^v - 1} = a_0 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\, a_\nu v^\nu,$$

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where $a_0 = q$,

$$a_\nu = D_v^\nu K_p\}_{v=0} = \frac{1}{\nu+1}\left[B\left(\frac{1+q}{2},\ \nu+1\right) - B\left(\frac{1-q}{2},\ \nu+1\right)\right]$$

$$= \begin{cases} \frac{2}{\nu+1} B\left(\frac{1+q}{2},\ \nu+1\right), & \nu \text{ even}, \\ 0, & \nu \text{ odd} \end{cases}$$

(Schlömilch, *Compendium*, II., p. 207, "The Bernoullian Functions B"). Hence the formula (1).

2. Since $(p-1)! \equiv (-1)^{\frac{1}{2}(p-1)}[1 \cdot 3 \cdot 5 \ldots (p-2)^2 \equiv -1 \pmod p$, (prime number), or

$$(-1)^{\frac{1}{2}(p-1)} r \equiv (-1)^{\frac{1}{2}(p-1)}[1 \cdot 3 \cdot 5 \ldots (p-2)^2] = mp-1,$$

where m of the form $4\mu+2$, we have

$$L_p = -\frac{\cos r\pi/2p}{\cos \pi/2p} = \begin{cases} 1, & p \text{ prime number} > 2, \\ 0, & p \text{ compound}, \end{cases}$$

$$\left[r = \{1 \cdot 3 \cdot 5 \ldots (p-2)^2\}\right];$$

$$L_p = -\frac{e^{i(1+r).\pi/2p} + e^{i(1-r).\pi/2p}}{e^{i\pi/p}+1} = -\frac{e^{(1+r)v}}{e^{2v}+1} - \frac{e^{(1-r)v}}{e^{2v}+1}$$

$$= -C_0 - \sum_{\nu=1}^{\infty} \frac{1}{\nu!} C_\nu v^\nu,$$

where $v = i \cdot \pi/2p$,

$$C_\nu = D_v^\nu L_p\}_{v=0} = D_v^\nu \frac{e^{(1+r)v}}{e^{2v}+1}\Big\}_0 + D_v^\nu \frac{e^{(1-r)v}}{e^{2v}+1}\Big\}_0$$

$$= \tfrac{1}{2}\left[E(r, \nu) + E(-r, \nu)\right] = \begin{cases} E(r, \nu), & \nu \text{ even}, \\ 0, & \nu \text{ odd}; \end{cases}$$

where E denotes the "Eulerian functions" of the first species, determined by Franz Rogel (*Sitzungsberichte der kgl. böhmischen Gesellschaft der Wissenschaften*, 1893, XXIII.); hence the formula (2). N.B.—"Eulerian functions" are not to be confounded with "Eulerian integrals."

3. Suppose p an odd prime number; then $(p-1)! = mp-1$ is divisible by 4; therefore mp is of the form $4\mu+1$; from this we deduce that m and p must be simultaneously either of the form $4\mu+1$ or $4\mu-1$. Therefore

$$(-1)^{\frac{1}{2}(m-1)} = (-1)^{\frac{1}{2}(p-1)},$$

and

$$\sin \frac{(p-1)!}{p} \tfrac{1}{2}\pi = (-1)^{\frac{1}{2}(p-1)} \cos \pi/2p,$$

or

$$M_p = (-1)^{p-1} \frac{\sin q/p \cdot \frac{1}{2}\pi}{\cos \pi/2p} = \begin{cases} 1, & p \text{ prime number} > 3, \\ 0, & p \text{ compound}. \end{cases}$$

Introducing exponential functions, we obtain

$$(-1)^{\frac{1}{2}(p-1)} M_p = -i\left\{\frac{e^{(1+q)v}}{e^{2v}+1} - \frac{e^{(1-q)v}}{e^{2v}+1}\right\} = -i\left(C_0 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} c_\nu v^\nu\right),$$

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where $v = i \cdot \pi/2p, \quad c_0 = 0,$

$$C_\nu = D_v^\nu \frac{e^{(1+q)v}}{e^{2v}+1}\Big\}_{v=0} - D_v^\nu \frac{e^{(1-q)v}}{e^{2v}+1}\Big\}_0$$

$$= \tfrac{1}{2}\left[E(q,\nu) - E(-q,\nu)\right] = \begin{cases} E(q,\nu), & \nu \text{ odd}, \\ 0, & \nu \text{ even}. \end{cases}$$

Hence the formula (3).

4. Setting $[\{\frac{1}{2}(p-1)\}!]^2 = s,$

we have $(p-1)! \equiv (-1)^{\frac{1}{2}(p-1)} \bmod p$ (prime number) $= mp-1$;

then $s^2 \equiv 0 \pmod 4$ for $p > 3$, and

$$N_p = \frac{\sin s/p \,.\, \frac{1}{2}\pi}{\cos \pi/2p} = \sum_{\nu=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(\nu-1)}}{\nu!} E(s,\nu)(\pi/2p)^\nu \text{ (formula 4)}$$

$$= \begin{cases} 1, & p \text{ prime number} > 3, \\ 0, & p \text{ compound}. \end{cases}$$

5. Denoting by $\mathfrak{N}_z$ the *number of prime numbers* $\leq z$, we have

$$\mathfrak{N}_z = \begin{cases} 2 + \sum_{p=3,5,\ldots}^{z} K_p, \\ 2 + \sum_{p=3,5,\ldots}^{z} L_p, \\ 3 + \sum_{p=5,7,\ldots}^{z} M_p, \\ 3 + \sum_{p=5,7,\ldots}^{z} N_p, \end{cases}$$

wherein for the factors of discontinuity K, L, M, N are to be substituted their equivalent series, given in (1), (2), (3), and (4).

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$$= \tfrac{1}{3}\Sigma_1^\infty \left(\frac{1}{12n-11} - \frac{1}{12n-5}\right) = \tfrac{1}{3}(1 - \tfrac{1}{7} + \tfrac{1}{13} - \tfrac{1}{19} + \ldots)$$

$$= \tfrac{1}{3}\Sigma\left\{1/\omega\,(\omega - \tfrac{1}{2}\omega^2 + \tfrac{1}{3}\omega^3 - \ldots)\right\} = \tfrac{1}{3}\Sigma\, 1/\omega \log(1+\omega)$$

(where ω has the values $e^{\pm i\pi/6}$, $e^{\pm i\pi/2}$, $e^{\pm 5i\pi/6}$)

$$= \tfrac{1}{3}\left\{\sqrt{3}\log(2+\sqrt{3}) + \pi\right\};$$

therefore sum to $2n+1$ terms when n is ∞

$$= \tfrac{1}{3} + \tfrac{1}{3}\left\{\sqrt{3}\log(2+\sqrt{3}) + \pi\right\}.$$

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